Mathematical Tools
for Neural and Cognitive Science

Fall semester, 2025

Section 1a: Trichromacy
(an extended linear algebra example)

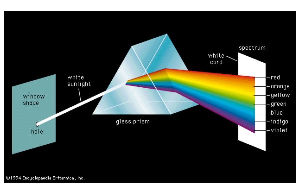
Trichromacy

A spectacular multi-disciplinary story …

- Physics: spectral nature and additivity of light [Newton, and others, 1600's]
- Perception: Trichromatic matching experiments [Grassman, Young, and others, 1850's]
- Mathematics: Human color matching is explained by a 3-dimensional linear model [late 1800's]
- Engineering: Devices for color reproduction require only three color channels [early 1930's]
- Neurobiology: Trichromacy in humans (and some other primates) arises from 3 cone types [late 1980's!]

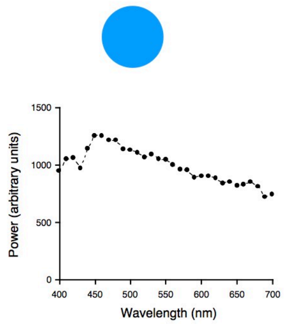
[See Wandell, *Foundations of Vision*, ch2]

Spectral nature of light



[Newton, 1665]

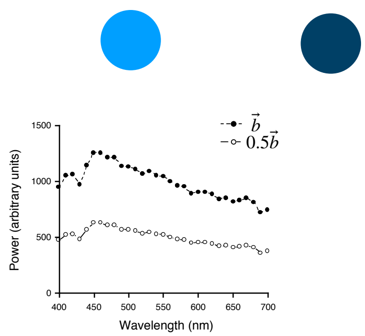
Representing spectra with vectors



$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \bullet \\ \bullet \\ \bullet \\ b_{31} \end{bmatrix}$$

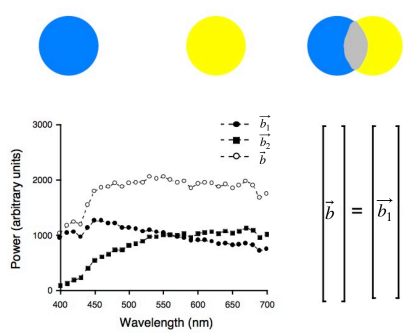
[c/o David Brainard]

Intensity adjustment (scalar multiplication)



[c/o David Brainard]

Additive mixtures



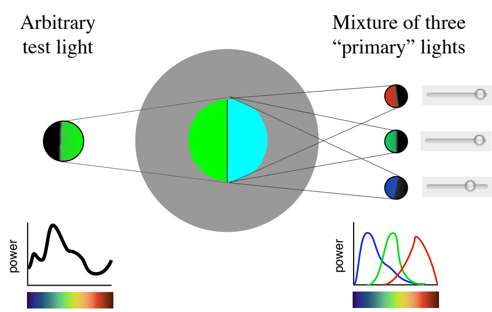
$$\begin{bmatrix} \\ \vec{b} \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \vec{b_1} \\ \\ \end{bmatrix} + \begin{bmatrix} \\ \vec{b_2} \\ \\ \end{bmatrix}$$

[c/o David Brainard]

Color perception: what's it for?

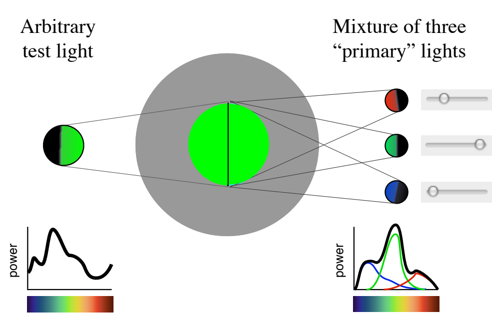
[c/o David Brainard]

Perceptual color matching experiment

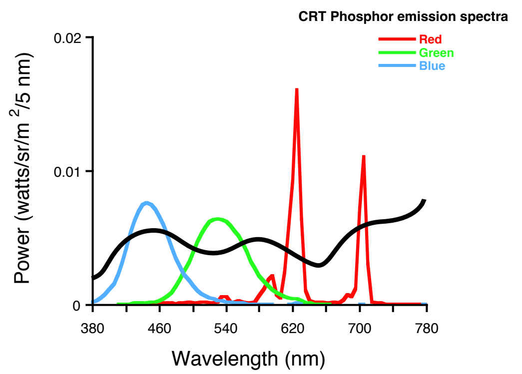


[Young, Helmholtz, Grassman, etc, 1800's]

Perceptual color matching experiment



[Young, Helmholtz, Grassman, etc, 1800's]



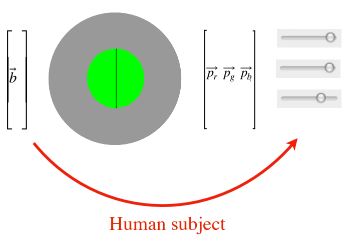
Grassmann's Laws (1853)

1) Any light can be matched with a mixture of 3 primaries
2) Rescaling the light results in a rescaled mixture
3) Adding two lights results in a sum of their mixtures

➡ Color matching can be described by an Nx3 linear system*

* Fine print: i) Normal human observer; ii) photopic intensity levels, not too bright, not too dim; iii) independent primaries.

Experimental measurement:



Linear model: $\begin{bmatrix}0.25\\0.75\\0.50\end{bmatrix} = \begin{bmatrix} & \mathbf{H} & \end{bmatrix} \begin{bmatrix}\vec{b}\end{bmatrix}$

[c/o David Brainard]

Implications

- If P is an $Nx3$ matrix containing the primary spectra, then the color matching experiment is captured by a $3xN$ "matching matrix" H that maps a light $\vec{l}$ to 3 "knob settings":

$$\vec{l} \sim PH\vec{l}$$

where $\sim$ means "looks the same as"

- Two lights look the same if (and *only* if) they produce the same match settings:

$$\vec{l}_1 \sim \vec{l}_2 \quad \Leftrightarrow \quad H\vec{l}_1 = H\vec{l}_2$$

Characterization

$$\text{400 nm} \quad \begin{bmatrix}1\\0\\0\\ \vdots\\0\end{bmatrix} \xrightarrow{H} \begin{bmatrix}9.76\\0.27\\46.4\end{bmatrix}$$

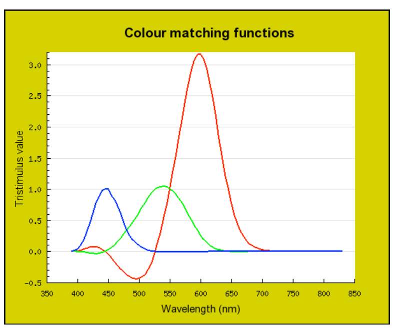
$$\text{410 nm} \quad \begin{bmatrix}0\\1\\0\\ \vdots\\0\end{bmatrix} \longrightarrow \begin{bmatrix}29.7\\0.82\\142\end{bmatrix}$$

$$\text{420 nm} \quad \begin{bmatrix}0\\0\\1\\ \vdots\\0\end{bmatrix} \longrightarrow \begin{bmatrix}91.8\\2.73\\440\end{bmatrix}$$

$$\cdots$$

$$\text{700 nm} \quad \begin{bmatrix}0\\0\\0\\ \vdots\\1\end{bmatrix} \longrightarrow \begin{bmatrix}7.79\\2.80\\0.00\end{bmatrix}$$

[c/o David Brainard]



Stiles-Burch Data, Monochromatic RGB Primaries of 645, 526, and 444 nm.

A matrix that mimics a human!

[Stiles & Burch, 1959]

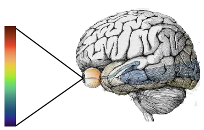
Useful calculations

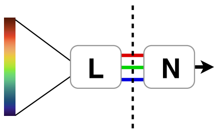
- Scientific: given results of one matching experiment, predict the results of another one (with different primaries).
- Practical: calibrate a display device, so as to generate mixtures of three colors that match the appearance of any desired real-world spectrum.

[derive on board]

Summary:

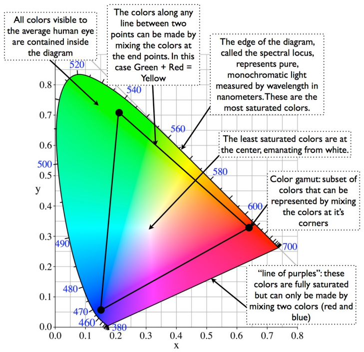
- Perceptual measurements: any light with can be matched with a mixture of 3 primaries
- Theory/model: the visual system projects the wavelength spectra of light onto a 3-dimensional subspace



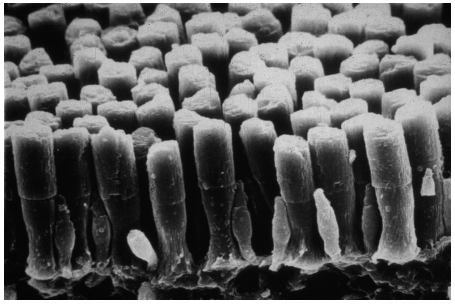


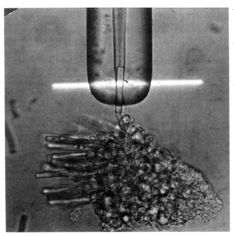
- Prediction/generalization: perceptual “metamers” - lights that appear identical, but have physically distinct wavelength spectra
- Engineering: CIE standards for color representation (1931).
- Underlying physiological mechanism (cones), verified in 1987!

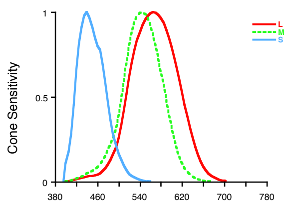
CIE Chromaticity Diagram



The underlying mechanism...

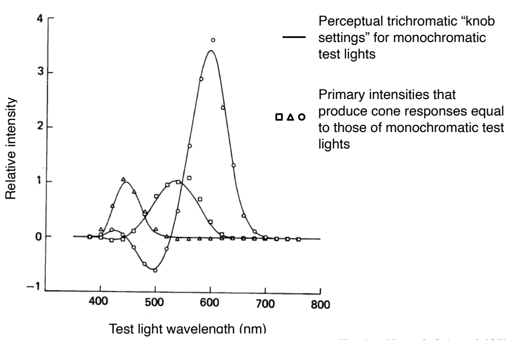






[Baylor, Nunn & Schnapf, 1987]

Confirmation of prediction (100+ years later!):



[Baylor, Nunn & Schnapf, '87]

Dimensionality caveats

- Normal human vision, at high ("photopic") light levels is tri-chromatic (cones)
- At low ("scotopic") light levels, we are all monochromats (rods only)
- At intermediate ("mesopic") light levels, we are quadrachromats (rods&cones)!
- Common genetic forms of color blindness are due to lack of one or two cone types