

Mathematical Tools for Neural and Cognitive Science

Fall semester, 2025

Section 1: Linear Algebra

Linear Algebra

“Linear algebra has become as basic and as applicable as calculus, and fortunately it is easier”

- Gilbert Strang, *Linear Algebra and its Applications*, 1980

... and this is even more true today than when the book was published!

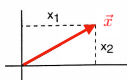
Vectors

Ordered lists of numbers,
depicted in 3 ways:

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_N \end{pmatrix}$$

In two or three dimensions, we can draw these as arrows:

N=2

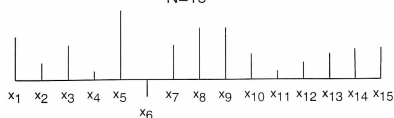


N=3



In higher dimensions, we typically must resort to a “spike-plot”:

N=15

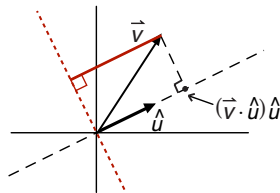


Vector operations

- scalar multiplication
- addition, vector spaces
- length (norm), unit vectors
- inner product (a.k.a. “dot” product)
 - definition/notation: sum of pairwise products
 - geometry: cosines, squared length, orthogonality test

[on board: geometry]

Inner product with a unit vector



- **projection** onto line
- **distance** to line/plane
- change of **coordinates**

[on board: geometry]

Vectors as “operators”

- “averager”
- “windowed averager”
- “smooth averager”
- “local differencer”
- “component selector”

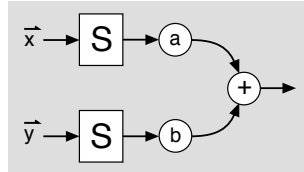
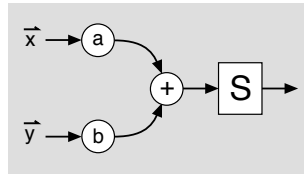
[on board]

Linear System

S is a linear system if (and only if) it obeys the **principle of superposition**:

$$S(a\vec{x} + b\vec{y}) = aS(\vec{x}) + bS(\vec{y})$$

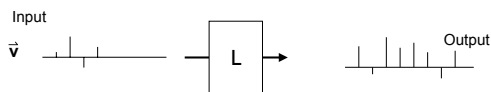
For *any* input vectors $\{\vec{x}, \vec{y}\}$, and *any* scalars $\{a, b\}$, the two diagrams at the right must produce the same response.



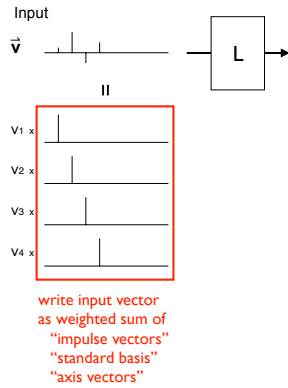
Linear Systems

- Very well understood (150+ years of effort)
- Excellent design/characterization toolbox
- An idealization (they do not exist!)
- Useful nevertheless:
 - conceptualize fundamental issues
 - provide baseline performance
 - building blocks for more complex models

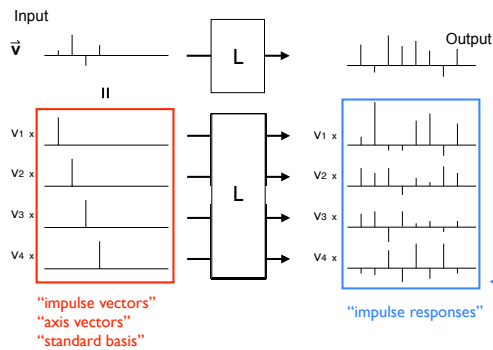
Implications of Linearity



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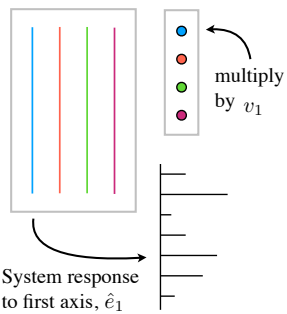


Response to *any* input is a weighted sum of responses to impulses
This defines the operation of *matrix multiplication*

Matrix multiplication

Two interpretations of $M\vec{v}$

input perspective:
weighted sum of columns



output perspective:
inner product with rows



[details on board]

Rafetto's Pasta (est. 1906)



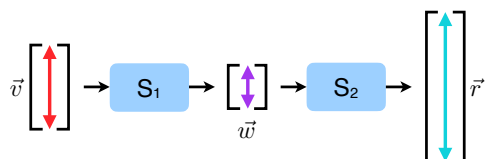
<https://raffettospasta.com>

Matrix multiplication

- two interpretations of $M\vec{v}$:
 - “input perspective”: weighted sum of columns
 - “output perspective”: inner product with rows
- transpose A^T , symmetric matrices ($A = A^T$)
- distributive property: directly from linearity!
- associative property: cascade of two linear systems is linear (defines matrix multiplication)

[details on board]

Cascaded linear systems => product of matrices

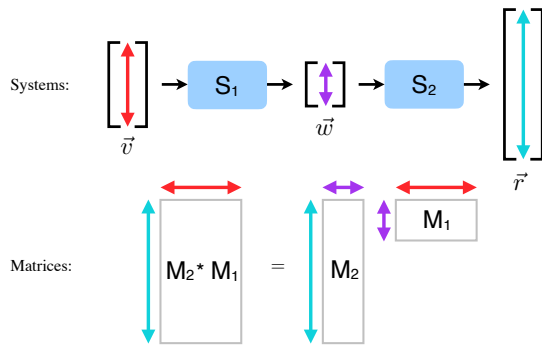


$$\vec{r} = M_2 M_1 \vec{v}$$

$$\vec{w} = M_1 \vec{v} = \sum_k v_k [M_1]_k \quad \vec{r} = M_2 \vec{w} = \sum_k v_k M_2 [M_1]_k$$

kth column of M_1

Cascaded linear systems => product of matrices



Math: $\vec{w} = M_1 \vec{v} = \sum_k v_k [M_1]_k$ $\vec{r} = M_2 \vec{w} = \sum_k v_k M_2 [M_1]_k$

Matrix multiplication

- two interpretations of $M\vec{v}$:
 - “input perspective”: weighted sum of columns
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- transpose A^T , symmetric matrices ($A = A^T$)
- distributive property: directly from linearity!
- associative property: cascade of two linear systems is linear. Defines matrix multiplication.
- generally *not* commutative ($AB \neq BA$), but note that $(AB)^T = B^T A^T$
- vectors as matrices. Inner products vs outer products

[details on board]

All matrices

Orthogonal matrices

- square shape (dimensionality-preserving)
- rows are orthogonal unit vectors
- columns are orthogonal unit vectors
- performs a rotation of the vector space (with possible axis inversion)
- preserves vector lengths and angles (and thus, dot products)
- inverse is transpose

Identity matrix
(w/ axis flips)

Diagonal matrices

- arbitrary rectangular shape
- all off-diagonal entries are zero
- squeeze/stretch along standard axes
- if non-square, creates/discards axes
- inverse is diagonal, with inverse of non-zero diagonal entries of original

(Venn Diagram)

Singular Value Decomposition (SVD)

Any matrix M can be factorized as

$$M = U S V^T$$

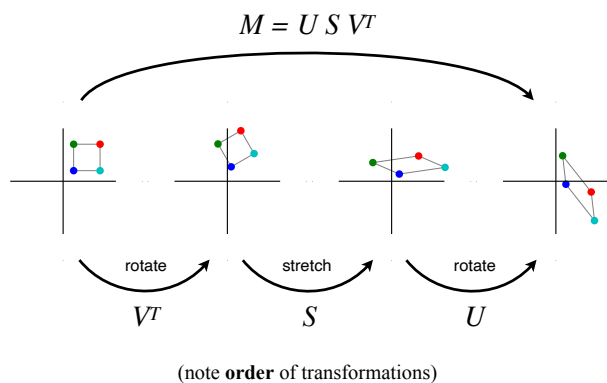
with U and V orthogonal, S diagonal

- geometry: “rotate, stretch, rotate”
- columns of V are basis for *input* coordinate system
- columns of U are basis for *output* coordinate system
- S rescales axes, and determines what “gets through”

[\[details on board\]](#)

SVD geometry (in 2D)

Apply M to four vectors (with heads at colored points):



Singular Value Decomposition (SVD)

Any matrix M can be factorized as

$$M = U S V^T$$

with U and V orthogonal, S diagonal

- unique (up to permutations and sign flips)
- sum of “outer products”
- nullspace and rangespace
- inverse and pseudo-inverse

[\[details on board\]](#)

