Mathematical Tools
for Neural and Cognitive Science

Fall semester, 2024

Section 1: Linear Algebra

Linear Algebra

"Linear algebra has become as basic and as applicable as calculus, and fortunately it is easier"

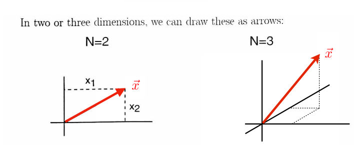
- Gilbert Strang, *Linear Algebra and its Applications*

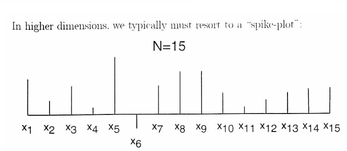
… and this is even more true today than when the book was published!

Vectors

Ordered lists of numbers, depicted in 3 ways:

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_N \end{pmatrix}$$



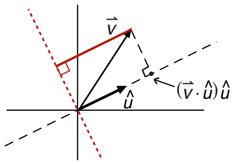


Vector operations

- scalar multiplication
- addition, vector spaces
- length (norm), unit vectors
- inner product (a.k.a. “dot” product)
  - definition/notation: sum of pairwise products
  - geometry: cosines, squared length, orthogonality test

[on board: geometry]

Inner product with a unit vector



- **projection** onto line
- **distance** to line/plane
- change of **coordinates**

[on board: geometry]

Vectors as “operators”

- “averager”
- “windowed averager”
- “smooth averager”
- “local differencer”
- “component selector”

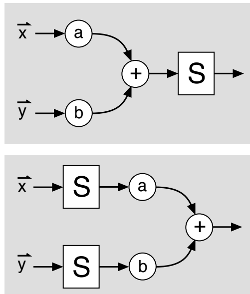
[on board]

Linear System

S is a linear system if (and only if) it obeys the principle of superposition:

$$S(a\vec{x} + b\vec{y}) = aS(\vec{x}) + bS(\vec{y})$$

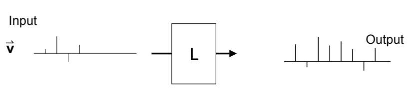
For *any* input vectors $\{\vec{x}, \vec{y}\}$, and *any* scalars $\{a, b\}$, the two diagrams at the right must produce the same response.



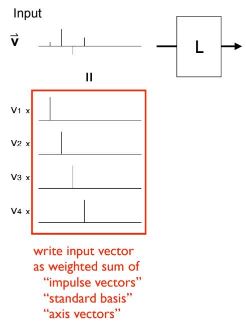
Linear Systems

- Very well understood (150+ years of effort)
- Excellent design/characterization toolbox
- An idealization (they do not exist!)
- Useful nevertheless:
  - conceptualize fundamental issues
  - provide baseline performance
  - building blocks for more complex models

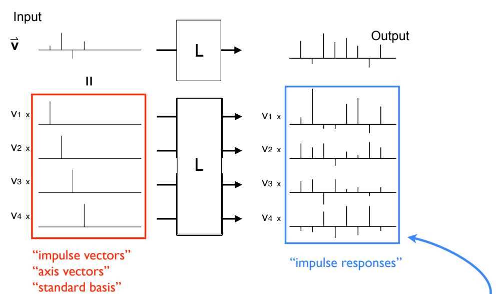
Implications of Linearity



Implications of Linearity



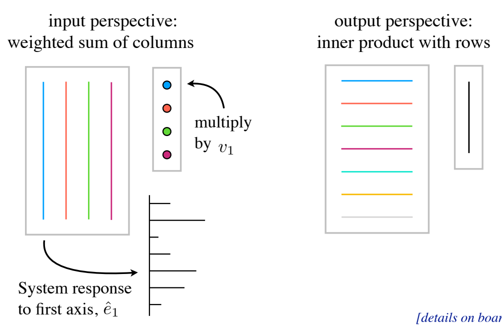
Implications of Linearity



Response to *any* input is a weighted sum of responses to impulses
This defines the operation of *matrix multiplication*

Matrix multiplication

Two interpretations of $M\vec{v}$



[details on board]

Rafetto's Pasta (est. 1906)

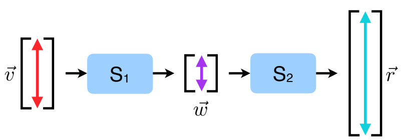
https://raffettospasta.com

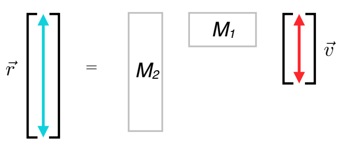
Matrix multiplication

- two interpretations of $M\vec{v}$:
  - weighted sum of columns
  - inner products with rows
- transpose: A^T, symmetric matrices ($A = A^T$)
- distributive property (directly from linearity)
- associative property: cascade of two linear systems is linear. Defines matrix multiplication.

[details on board]

Cascaded linear systems => product of matrices

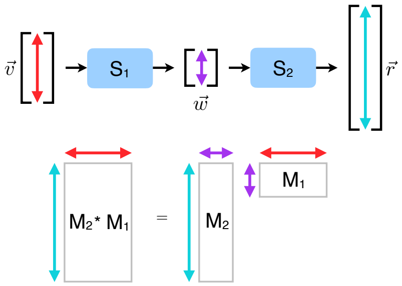




$$\vec{w} = M_1\vec{v} = \sum_k v_k [M_1]_k \qquad \vec{r} = M_2\vec{w} = \sum_k v_k M_2 [M_1]_k$$

kth column of M_1

Cascaded linear systems => product of matrices



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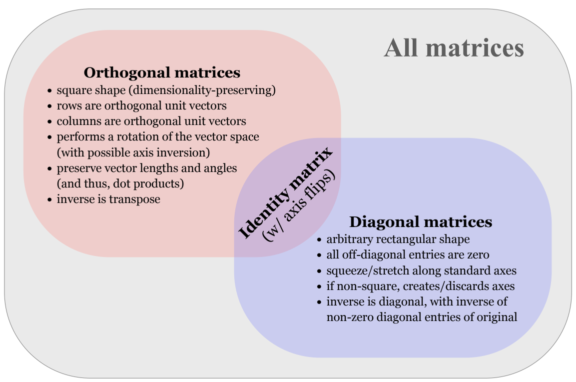
Matrix multiplication

- two interpretations of $M\vec{v}$:
  - "input perspective": weighted sum of columns
  - "output perspective": inner product with rows
- transpose A^T, symmetric matrices ($A = A^T$)
- distributive property: directly from linearity!
- associative property: cascade of two linear systems is linear. Defines matrix multiplication.
- generally *not* commutative ($AB \neq BA$), but note that $(AB)^T = B^T A^T$
- vectors as matrices. Inner products vs outer products

[details on board]



(Venn Diagram)

Singular Value Decomposition (SVD)

Any matrix M can be factorized as

$$M = U\,S\,V^T$$

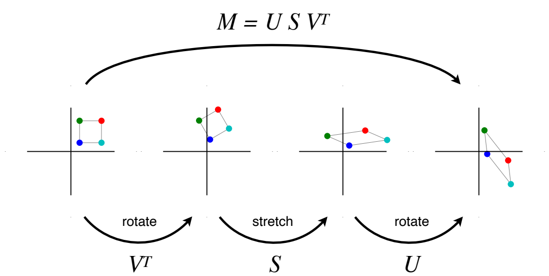
with U, V orthogonal, S diagonal

- geometry: "rotate, stretch, rotate"
- columns of V are basis for *input* coordinate system
- columns of U are basis for *output* coordinate system
- S rescales axes, and determines what "gets through"

[details on board]

SVD geometry (in 2D)

Apply M to four vectors (heads at colored points):



(note order of transformations)

Singular Value Decomposition (SVD)

Any matrix M can be factorized as

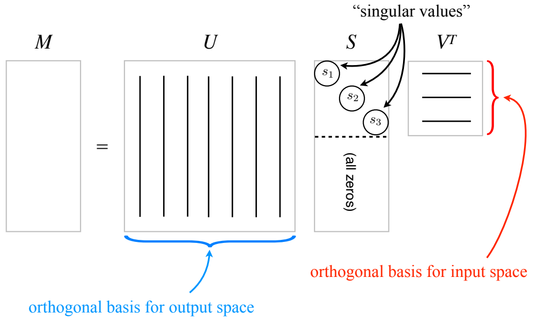
$$M = U\,S\,V^T$$

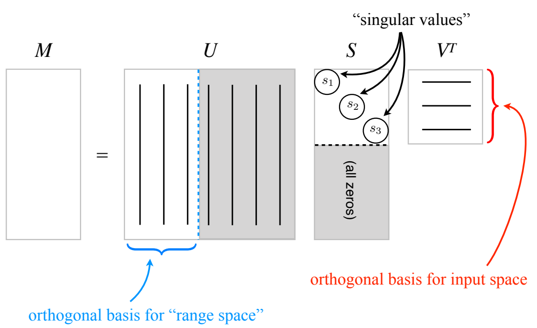
with U, V orthogonal, S diagonal

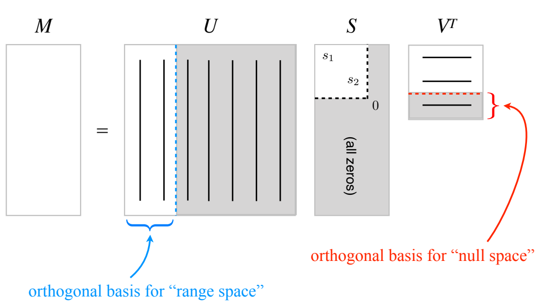
- unique, up to permutations and sign flips
- sum of "outer products"
- nullspace and rangespace
- inverse and pseudo-inverse

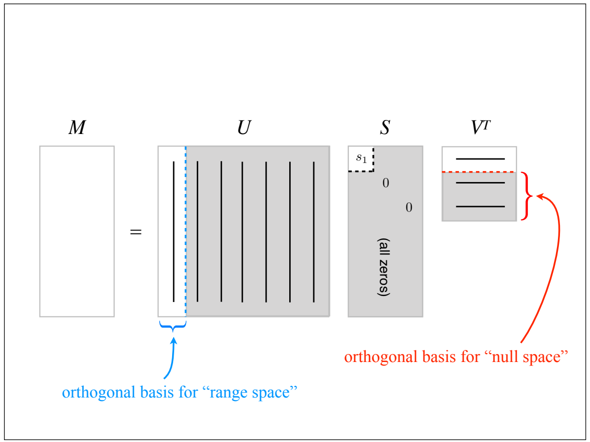
[details on board]

$$M\vec{x} = \sum_k \hat{u}_k \left(s_k \left(\hat{v}_k^T \vec{x}\right)\right) = \sum_k s_k \left(\hat{u}_k \hat{v}_k^T\right) \vec{x}$$ (sum of outer products)









M
U
S
V^T
=
s_1
0
0
(all zeros)
orthogonal basis for "range space"
orthogonal basis for "null space"