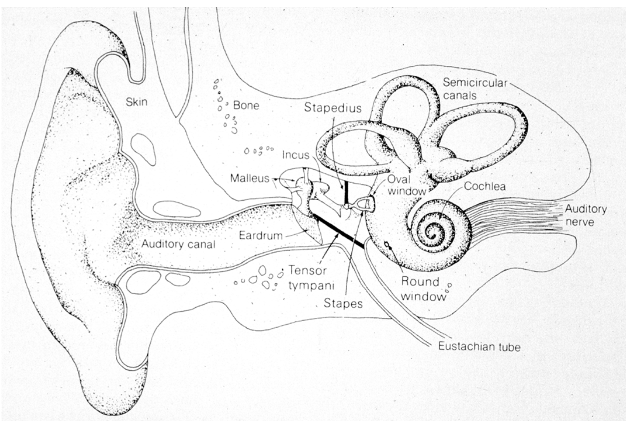
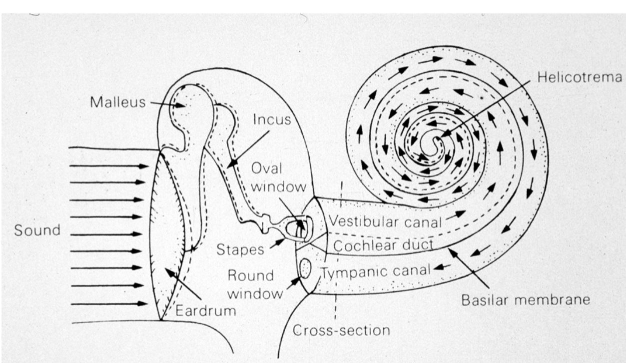
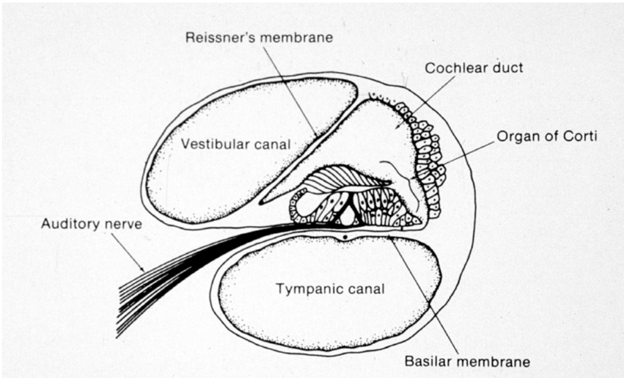
Mathematical Tools
for Neural and Cognitive Science

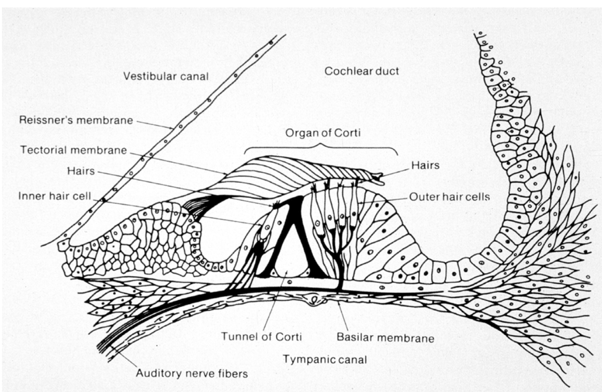
Fall semester, 2021

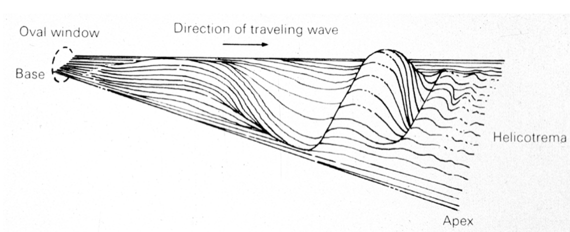
Section 3a: Early auditory system (an extended LSI example)

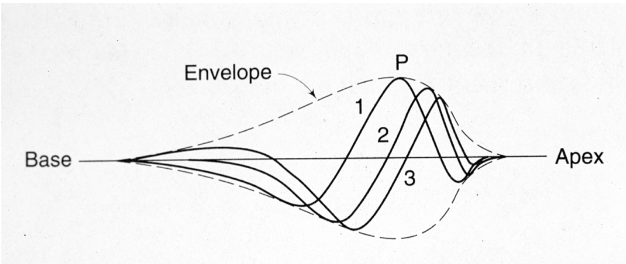


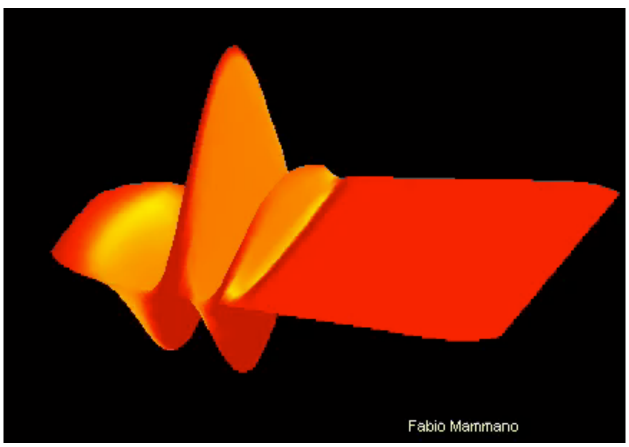


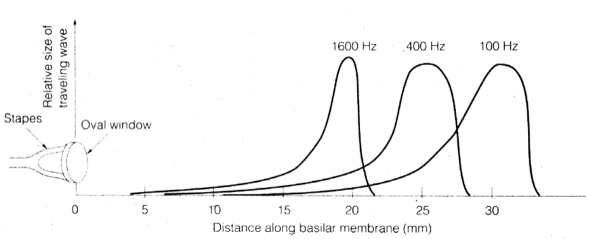

Reissner's membrane
Cochlear duct
Vestibular canal
Organ of Corti
Auditory nerve
Tympanic canal
Basilar membrane

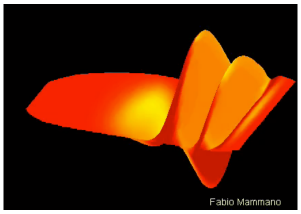

Vestibular canal
Cochlear duct
Reissner's membrane
Organ of Corti
Tectorial membrane
Hairs
Hairs
Inner hair cell
Outer hair cells
Tunnel of Corti
Basilar membrane
Tympanic canal
Auditory nerve fibers


Oval window
Direction of traveling wave
Base
Helicotrema
Apex

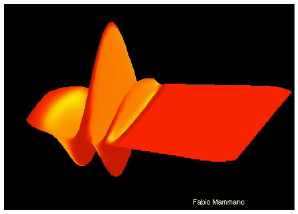

Envelope
P
1
2
3
Base
Apex


Fabio Mammano


Relative size of traveling wave
1600 Hz
400 Hz
100 Hz
Stapes
Oval window
0
5
10
15
20
25
30
Distance along basilar membrane (mm)



400 Hz



4000 Hz

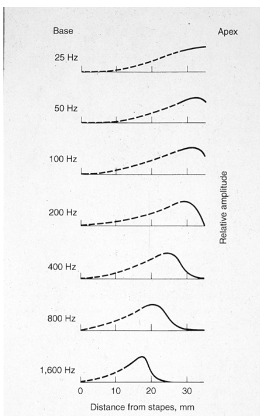
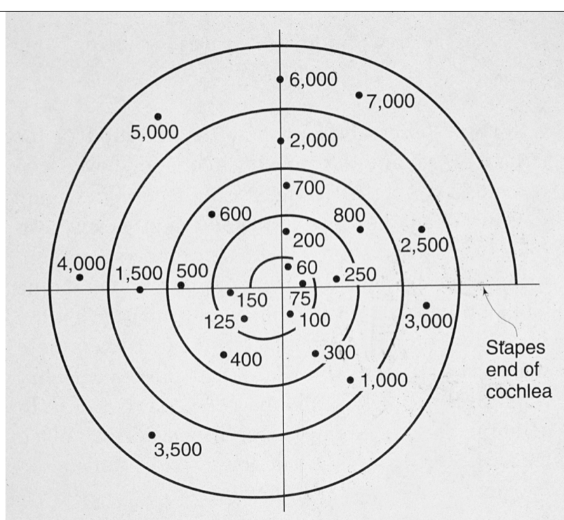


Figure 11.28
The envelope of the basilar membrane's vibration at frequencies ranging from 25 to 1,600 Hz, as measured by Bekesy (1960).



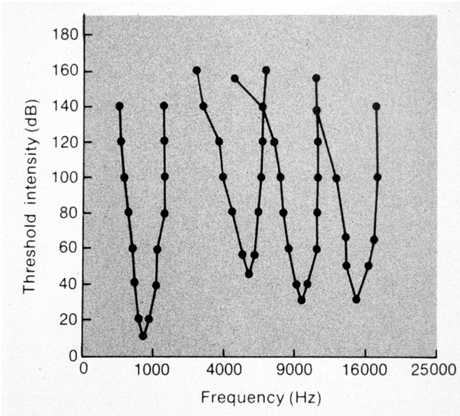
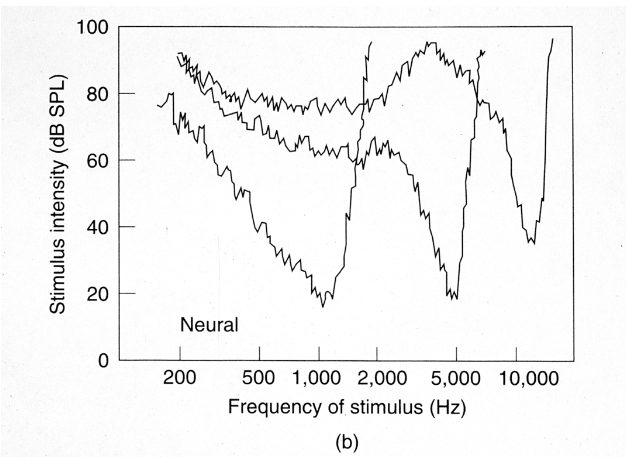
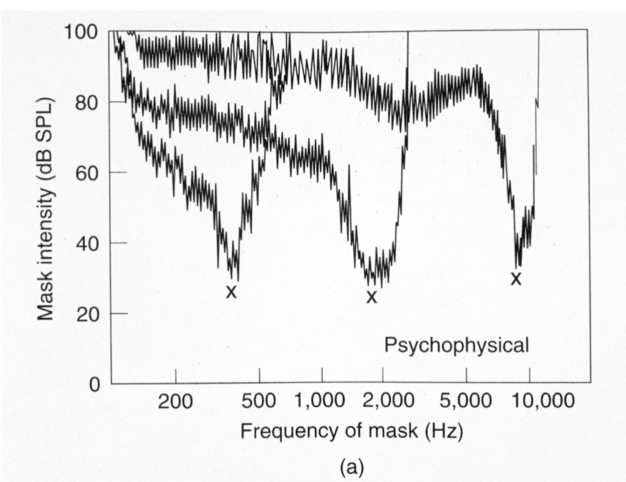


Figure 6–14 Threshold response curves for auditory nerve fibers in the cat (based on Whitfield, 1968).

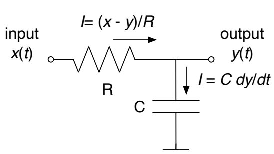


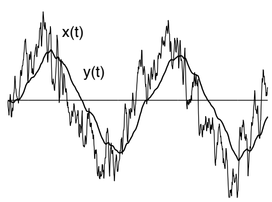


Critical Bands

- Loudness summation
- Critical band masking

"Designing" Filters





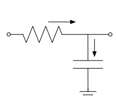
$$y(t) = V_C = \frac{1}{C}\int I\,dt \quad \text{or} \quad I = C\frac{dV_C}{dt}$$

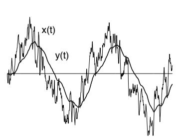
$$x(t) - y(t) = V_R = IR$$

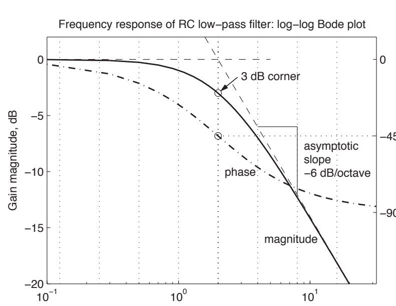
$$x - y = RC\frac{dy}{dt} = \tau\frac{dy}{dt}$$

For $x = 0$, solutions have the form $y = Ae^{-t/\tau}$

"Designing" Filters

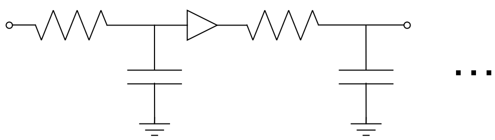






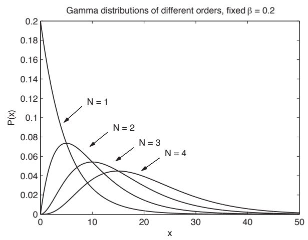
$$\tilde{y}(\omega) = \frac{1}{i\tau\omega + 1}$$

Cascaded or Gamma Filters



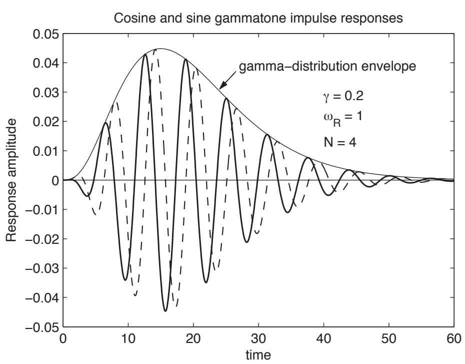
$$y(n) \propto t^{n-1} e^{-t/\tau}$$

$$\tilde{y}(\omega) = \left(\frac{1}{i\tau\omega + 1} \right)^{n}$$

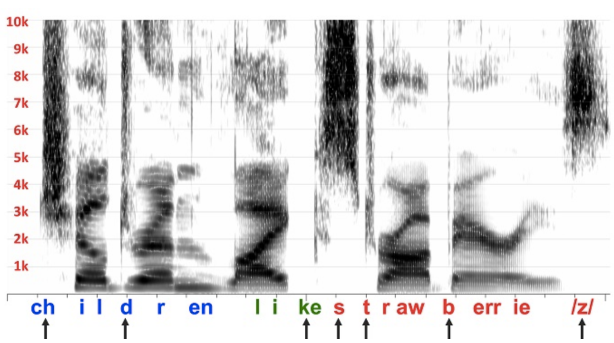


Gammatone Filters

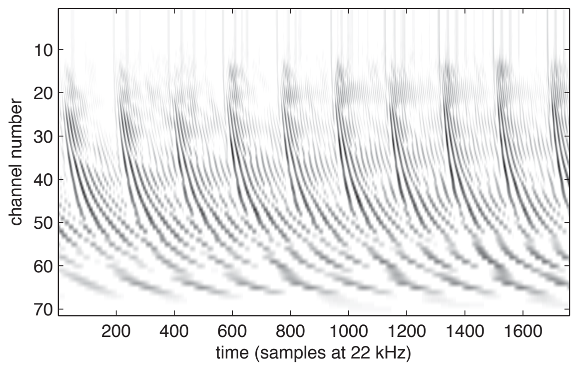
$$y \propto t^{n-1} e^{-t/\tau} \cos(\omega t - \phi)$$



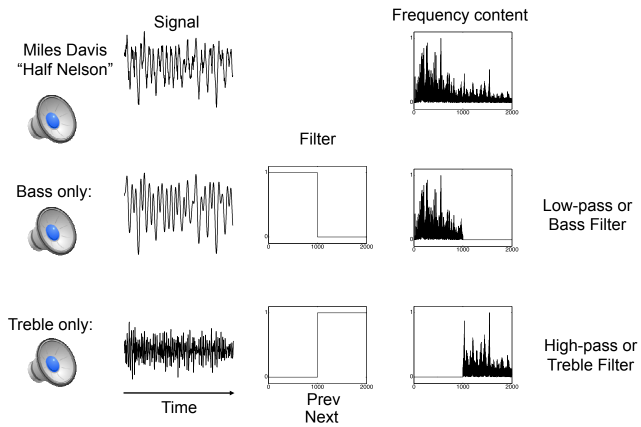
Speech Spectrogram



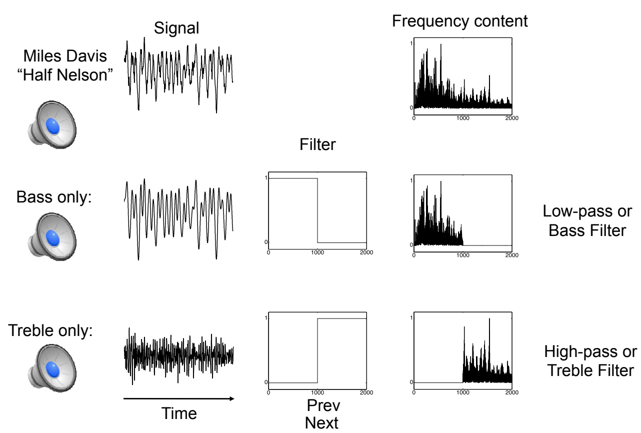
Cochleogram



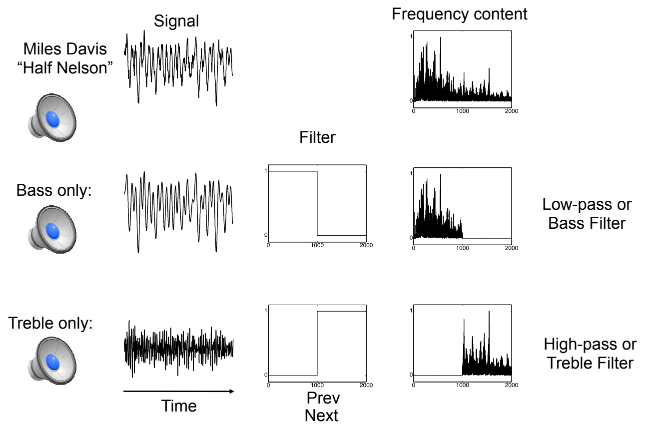
Example – Bass/Treble filters



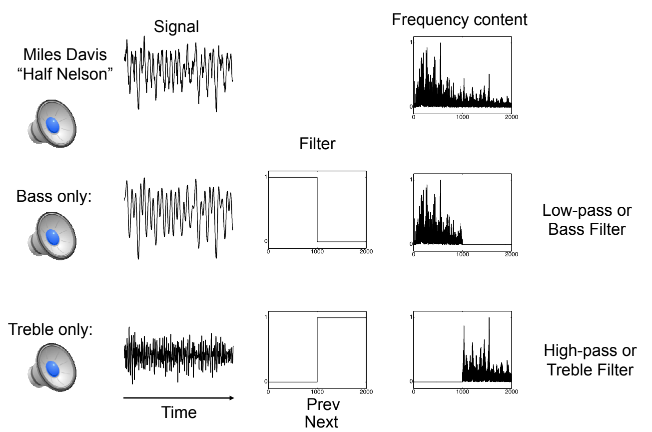
Example – Bass/Treble filters



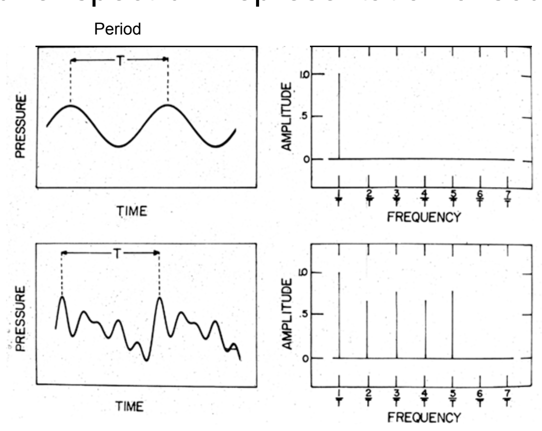
Example – Bass/Treble filters



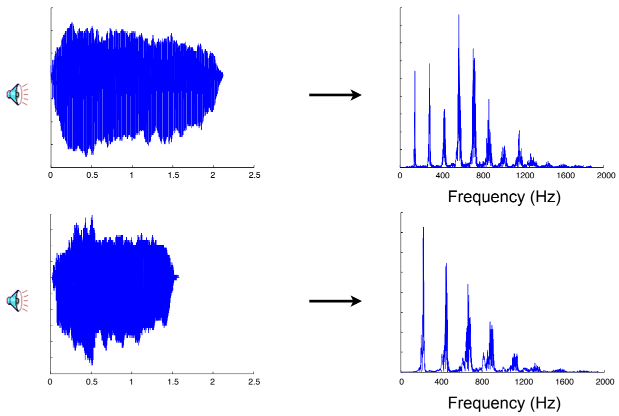
Example – Bass/Treble filters



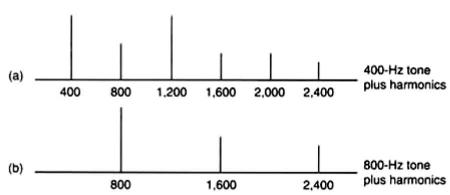
Fourier spectrum representation of sound



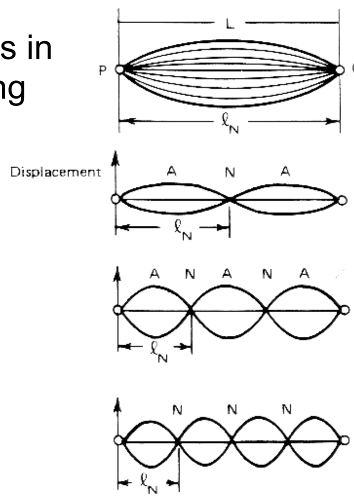
This works for all “musical sounds”



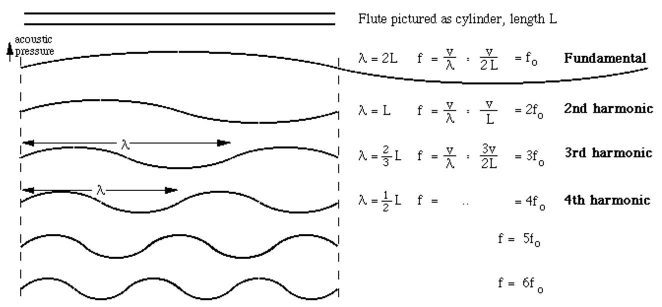
Fundamental frequency and harmonics



Standing waves in a vibrating string



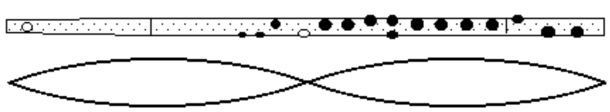
Flute (open pipe) harmonics



Flute (open pipe) harmonics

Other notes (shorten the pipe)

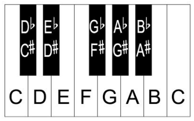
Reinforcing a harmonic (forcing a "node")



Musical temperament

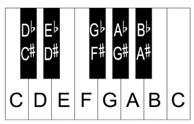
- How should you tune your piano, i.e., what "fundamental" frequency should be associated with each note?
- Equivalently: Where should the guitar frets be placed and strings be tuned? Where should the air holes on the flute be cut?
- http://virtualpiano.net

Musical basics

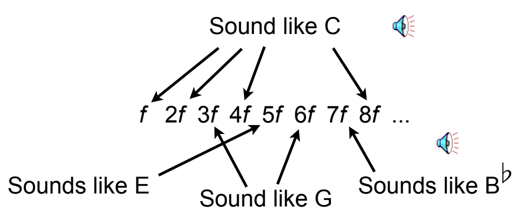


- The western scale has 12 notes: C, C#, D, D#, E, F, F#, G, G#, A, A#, B
- Half steps up from one note are the same ("enharmonic") as half steps down from a higher note. For example, C# = D♭ ("C sharp equals D flat")
- Perception of pitch is "circular", so that after B, then next higher note is C yet again, described as one "octave" above the next lower note named C
- Perception of pitch depends on ratios. In particular, an increase of one octave, such as from one C to the next higher C, DOUBLES the frequency (C = 256 Hz, next C = 512 Hz).

Musical harmonics

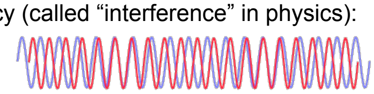


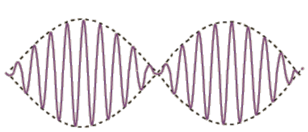
- An instrument playing C (frequency f = 256 Hz) will generally also produce harmonics $2f$, $3f$, $4f$, ...
- What notes do these harmonics sound like when heard on their own?



Aside: Beat frequencies

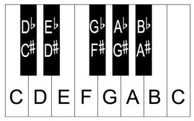
- What happens when unison notes are mistuned?
- $$\cos\left(2\pi\left(f-\frac{\Delta f}{2}\right)t\right)+\cos\left(2\pi\left(f+\frac{\Delta f}{2}\right)t\right)=2\cos\left(2\pi ft\right)\cos\left(2\pi\Delta ft\right)$$
- That is, you here the average frequency "beating", i.e., modulating up and down in volume, at the "difference" frequency (called "interference" in physics):

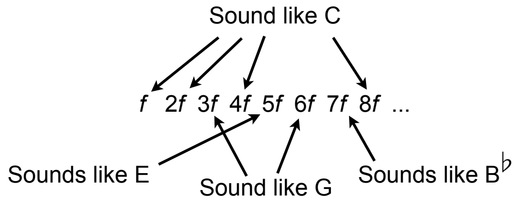




256 Hz + 258 Hz

Musical harmonics



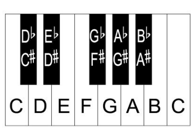


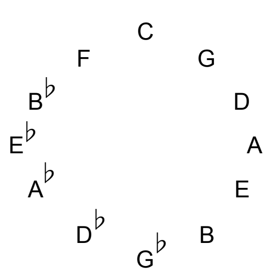
Thus, the most “consonant” (pleasant sounding, as opposed to “dissonant”) interval other than the octave (which sounds like unison) is the “perfect fifth” from C to G (seven half-steps), with a frequency ratio of 3 (or 3/2, 3/4, 6, 12, ...), because then when C & G are played together, the harmonics don’t “beat”.

Musical temperament, contd.

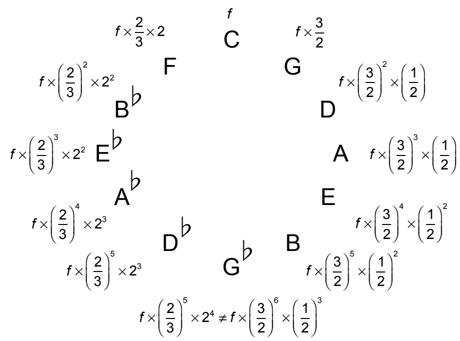
- A tuning system is a choice of what frequencies (or frequency ratios) correspond to the notes: C, C#, D, D#, E, F, F#, G, G#, A, A#, B.
- A “just” temperament tries, when possible, to have the frequency ratios be simple fractions, such as 3/2 for G relative to C.

The circle of 5ths





The Pythagorean scale (yes, that Pythagoras, sort of)



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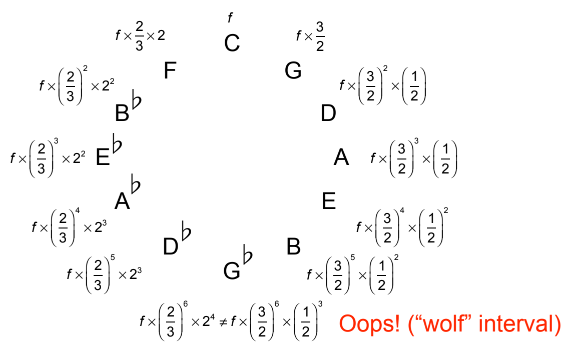
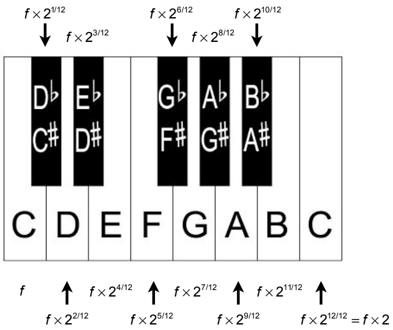


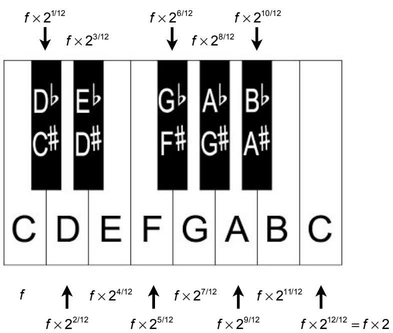
Musical temperament, contd.

- Thus, a just temperament, based on fractions, cannot keep the intended “perfect” interval (e.g., 3/2 for the interval of a 5th) in every key.
- This leads to keys with so-called wolf intervals, making them unusable.
- There are many different just temperaments.
- By Bach’s time, Western music did not use Pythagorean tuning, but rather something called “quarter-comma meantone”, which is based on making most of the Major 3rds have a ratio of 5/4, but also has several wolf intervals.
- How can you split the octave (12 semitones) equally?

Equal temperament



Equal temperament



This works (mathematically, if not musically)!

Comparing temperaments

- Tuning systems are based on frequency ratios.
- Ratios are best described logarithmically, because of the property that log turns multiplication into addition:
  $\log(ab) = \log(a) + \log(b)$
  so that going up by two musical intervals (a product of two ratios) looks like addition.
- The usual unit used for this is the "cent", with 100 cents per chromatic step and 1200 cents per octave, so that an interval from f_1 to f_2, in cents, is $1200 \log_2(f_2 / f_1)$

Comparing temperaments

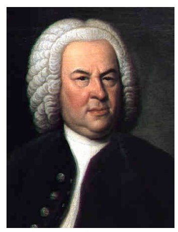
	C	C#	D	D#	E	F	F#	G	G#	A	A#	B	C
Equal	0	100	200	300	400	500	600	700	800	900	1000	1100	1200
Pythagorean	0	90	204	294	408	498	612	702	792	906	996	1110	1200
Equal 5th	700	700	700	700	700	700	700	700	700	700	700	700	700
Pythagorean 5th	702	702	702	702	702	702	678	702	702	702	702	702	702

What does this have to do with Bach?

The Well-Tempered Clavier (first book: 1722) contains one prelude and one fugue in each of the 12 Major and minor keys.

The point is: In equal temperament, and in well-designed (i.e., “well tempered”) unequal temperaments, you can play all 24 pieces in a row without retuning your clavier.

The fact that Bach used the term “well tempered” almost certainly means he was *not* using equal temperament at the time. The term comes from the writings of a contemporary, Werckmeister, and likely he was using a tuning suggested by Werckmeister.



J. S. Bach
(1685-1750)

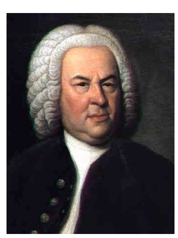
The subtleties

Bach’s prelude (BWV 998) for lute or clavier:

As I said, Bach likely was NOT using equal temperament but, rather, an unequal temperament from a contemporary (a system called Werckmeister III) that allows one to play in all keys, but gives each key a different character. Here is Bach’s Chromatic Fantasy for harpsichord:

Equal temperament (Christophe Rousset)

Werckmeister III (Rebecca Pechefsky)



J. S. Bach
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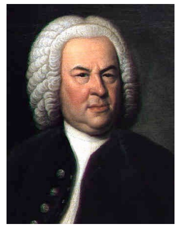
The subtleties



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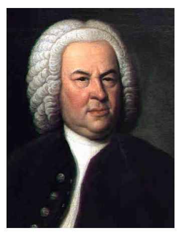

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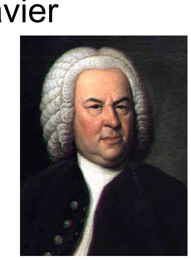
Werckmeister III (Rebecca Pechefsky)

The Well-Tempered Clavier



J. S. Bach
(1685-1750)

Thus, The Well-Tempered Clavier (Book One: 1722; Book Two: 1742) are two books of 24 preludes and 24 fugues, one pair in each of the 12 major and 12 minor keys. It effectively serves as an advertisement for "well temperament" (not likely equal temperament, and research suggests he used Werckmeister III).

Here is Rebecca Pechefsky performing Book One, Prelude No. 1 in C Major, on harpsichord, tuned in Werckmeister III.