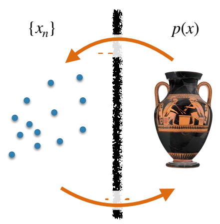
Mathematical Tools
for Neural and Cognitive Science

Fall semester, 2023

Section 5:
Statistical Inference and Model Fitting

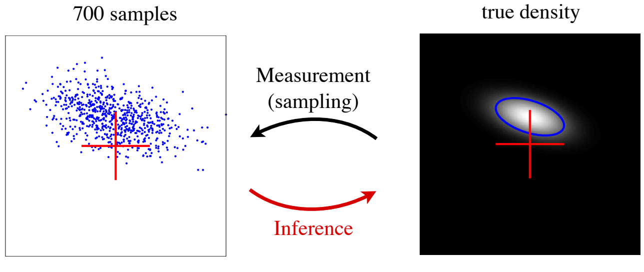
The sample average



$$\bar{x} = \frac{1}{N}\sum_{n=1}^{N} x_n$$

What happens as N grows?

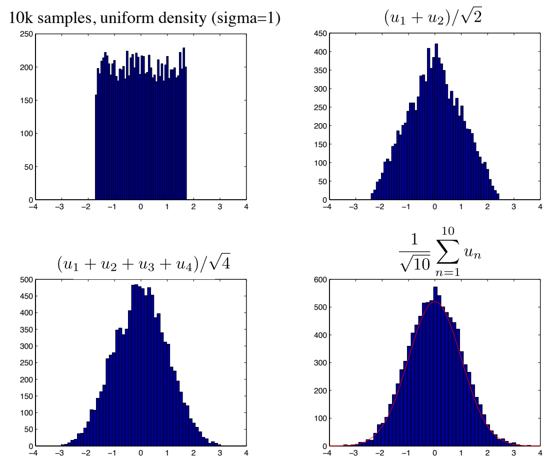
- Variance of $\bar{x}$ is σ_x^2/N (the "standard error of the mean", or SEM), and so converges to zero *[on board]*
- "Unbiased": $\bar{x}$ converges to the true mean, $\mu_x = \mathbb{E}(\bar{x})$ (formally, the "law of large numbers") *[on board]*
- The distribution $p(\bar{x})$ converges to a Gaussian (mean μ_x and variance σ_x^2/N): formally, the "Central Limit Theorem"



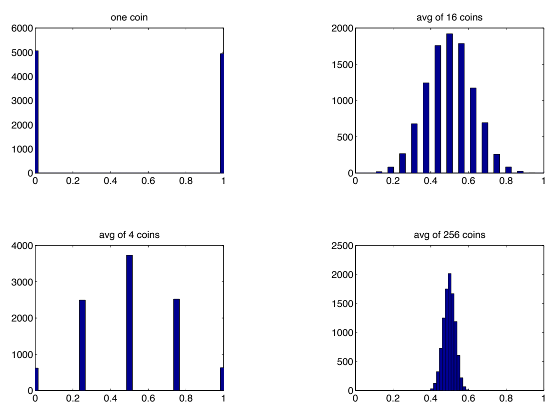
sample mean: [-0.05 0.83]
sample cov: [0.95 -0.23
-0.23 0.29]

true mean: [0 0.8]
true cov: [1.0 -0.25
-0.25 0.3]

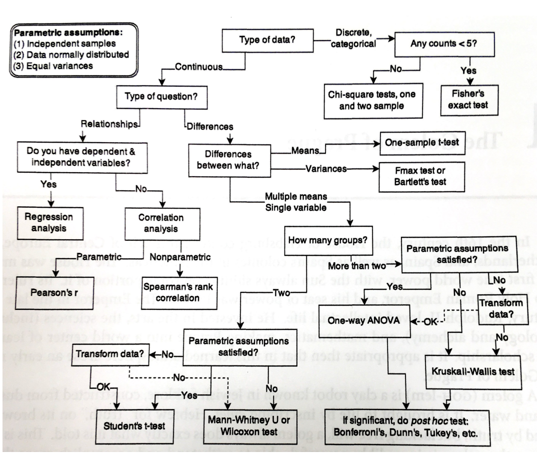
Central limit for a uniform distribution...



Central limit for a binary distribution...



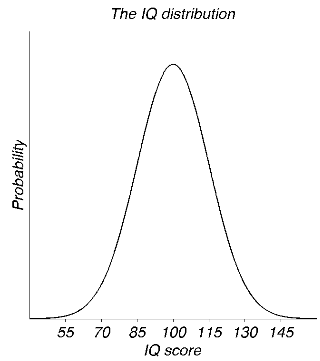
Classical "frequentist" statistical tests



Statistical Rethinking, Richard McElreath

Classical/frequentist approach - z

- In the general population, IQ is known to be distributed normally with
  - $\mu = 100$, $\sigma = 15$
- We give a drug to 30 people and test their IQ
- H_1: NZT improves IQ
- H_0 ("null"): it does nothing



Test statistic

- We calculate how far the observed value of the sample average is away from its expected value.
- In units of standard error.
- In this case, the test statistic is

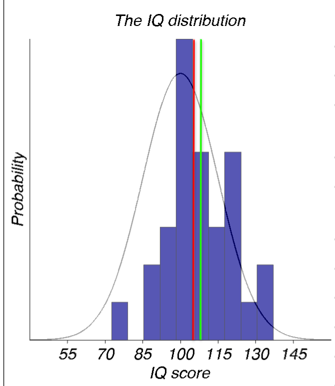
$$z = \frac{\bar{x} - \mu}{SE} = \frac{\bar{x} - \mu}{\sigma / \sqrt{N}}$$

- Compare to a distribution, in this case z or $N(0,1)$

Does NZT improve IQ scores or not?

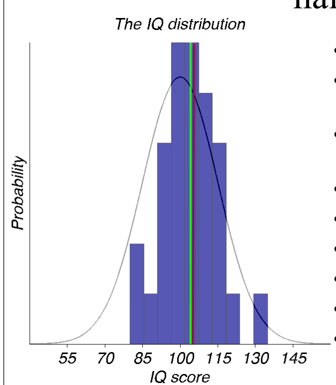
Decision \ Reality	Yes	No
Yes	Correct	Type I error α-error "False alarm"
No	Type II error β-error "Miss"	Correct

The z-test



- μ = 100 (Population mean)
- σ = 15 (Population standard deviation)
- N = 30 (Sample contains scores from 30 participants)
- $\bar{x}$ = 108.3 (Sample mean)
- $z = (\bar{x} - \mu)/\text{SE} = (108.3\text{-}100)/\text{SE}$ (Standardized score)
- $\text{SE} = \sigma / \sqrt{N} = 15/\sqrt{30} = 2.74$
- Error bar/CI: ±2 SE
- $z = 8.3/2.74 = 3.03$
- $p = 0.0012$
- Significant?
- One- vs. two-tailed test

What if the measured effect of NZT had been half that?



- μ = 100 (Population mean)
- σ = 15 (Population standard deviation)
- N = 30 (Sample contains scores from 30 participants)
- $\bar{x}$ = 104.2 (Sample mean)
- $z = (\bar{x} - \mu)/\text{SE} = (104.2\text{-}100)/\text{SE}$
- $\text{SE} = \sigma / \sqrt{N} = 15/\sqrt{30} = 2.74$
- $z = 4.2/2.74 = 1.53$
- $p = 0.061$
- Significant?

Significance levels

- Are denoted by the Greek letter α.
- In principle, we can pick anything that we consider unlikely.
- In practice, the consensus is that a level of 0.05 or 1 in 20 is considered as unlikely enough to reject H_0 and accept the alternative.
- A level of 0.01 or 1 in 100 is considered “highly significant” or “really unlikely”.

Common misconceptions

Is "Statistically significant" a synonym for:

- Substantial
- Important
- Big
- Real

Does statistical significance gives the

- probability that the null hypothesis is true
- probability that the null hypothesis is false
- probability that the alternative hypothesis is true
- probability that the alternative hypothesis is false

Meaning of *p*-value. Meaning of CI.

Student's *t*-test

- σ not assumed known
- Use

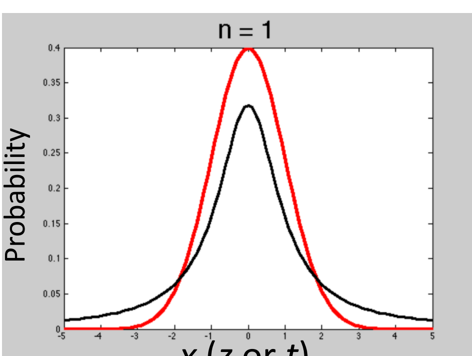
$$s^2 = \frac{\sum_{i=1}^{N}\left(x_i - \overline{x}\right)^2}{N-1}$$

- Why N-1? s is unbiased (unlike ML version), i.e., $\mathbb{E}(s^2) = \sigma^2$
- Test statistic is

$$t = \frac{\overline{x} - \mu_0}{s / \sqrt{N}}$$

- Compare to t distribution for CIs and NHST
- "Degrees of freedom" reduced by 1 to N-1

The *t* distribution approaches the normal distribution for large *N*



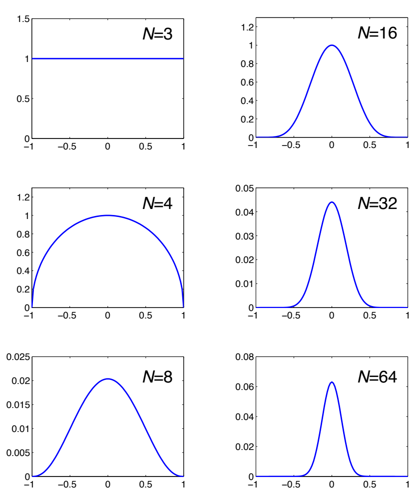
The z-test for binomial data

- Is the coin fair?
- Lean on central limit theorem
- Sample is n heads out of m tosses
- Sample mean: $\hat{p} = n / m$
- H_0: $p = 0.5$
- Binomial variability (one toss): $\sigma = \sqrt{pq}$, where $q = 1 - p$
- Test statistic: $$z = \frac{\hat{p} - p_0}{\sqrt{p_0 q_0 / m}}$$
- Compare to z (standard normal)
- For CI, use $$\pm z_{\alpha/2} \sqrt{\hat{p}\hat{q} / m}$$

Other frequentist univariate tests

- χ^2 goodness of fit
- χ^2 test of independence
- test a variance using χ^2
- F to compare variances (as a ratio)
- Nonparametric tests (e.g., sign, rank-order, etc.)

Lack of correlation is favored in N>3 dimensions



Null Hypothesis: Distribution of normalized dot product of pairs of Gaussian random vectors in N dimensions:

$$(1 - d^2)^{\frac{N-3}{2}}$$

Estimation of model parameters (outline)

- How do I compute estimates from data?
- How "good" are my estimates?
- How well does my model explain data to which it was fit? Other data (prediction/generalization)?
- How do I compare models?

Estimation

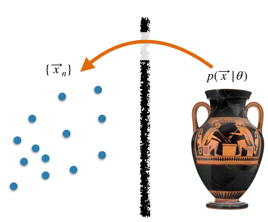
- An "estimator" is a function of the data, intended to provide an approximation of the "true" value of a parameter
- One can evaluate estimator quality in terms of squared error, MSE = bias^2 + variance
- Traditional statistics often aims for an unbiased estimator, with minimal variance ("MVUE")
- More nuanced view: trade off bias and variance, through model selection, "regularization", or Bayesian "priors"

The maximum likelihood (ML) estimator

Sample average is appropriate when one has direct measurements of the thing being estimated. But one may want to estimate something (e.g., a model parameter) that is *indirectly* related to the measurements…

Natural choice: assuming a probability model $p(\vec{x}\,|\,\theta)$ find the value of θ that maximizes this "likelihood" function

$$\hat{\theta}(\{\vec{x}_n\}) = \arg\max_{\theta} \prod_n p(\vec{x}_n|\theta)$$
$$= \arg\max_{\theta} \sum_n \log p(\vec{x}_n|\theta)$$



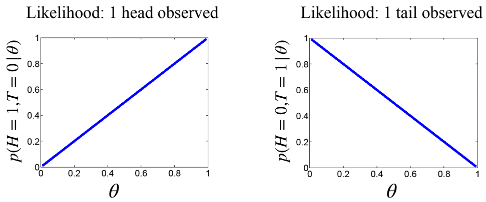
Example: Estimate the probability of a flipped coin landing “heads” up, by observing some samples

Example ML Estimators - discrete

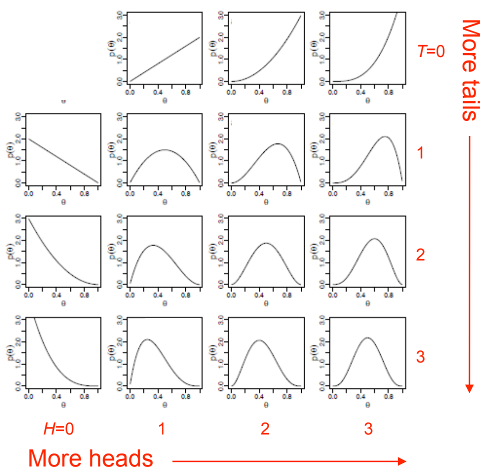
Binomial: $p(H \mid N, \theta) = \begin{pmatrix} N \\ H \end{pmatrix} \theta^{H} (1-\theta)^{N-H}$

(H = # heads observed, in N flips of a coin, with probability of heads θ)

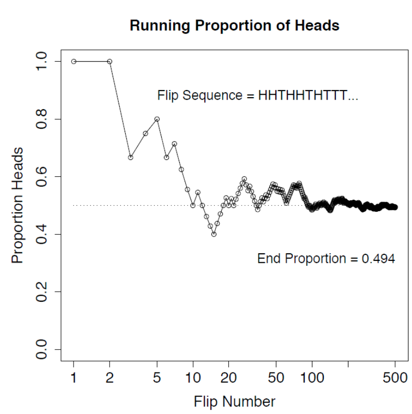
$$\hat{\theta}_{\text{ML}} = \frac{H}{N}$$



Likelihoods, $p(H \mid N, \theta)$



Convergence ("consistency")



Example ML Estimators - discrete

Binomial: $p(H \mid N,\theta) = \binom{N}{H} \theta^{H} (1-\theta)^{N-H}$

(H = # heads observed, in N flips of a coin, with probability of heads θ)

$$\hat{\theta}_{\text{ML}} = \frac{H}{N}$$

Poisson: $p(\{k_n\} \mid \theta) = \prod_{n=1}^{N} \frac{\theta^{k_n} e^{-\theta}}{k_n!}$

(k's are measured counts, with mean arrival rate of θ)

$$\hat{\theta}_{\text{ML}} = \frac{1}{N} \sum_{n=1}^{N} k_n$$

[on board]

Example ML Estimators - Continuous

Uniform: $p(x|\theta) = \begin{cases} \frac{1}{\theta} & 0 \le x \le \theta \\ 0 & \text{otherwise} \end{cases}$

$$\hat{\theta}_{\text{ML}} = \max_n \{x_n\}$$

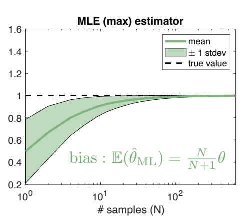
(Note: this is biased!)

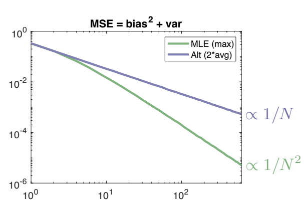
Two estimators for range of a uniform distribution

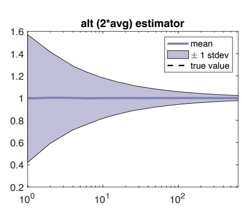
Given N samples $\{x_n\}$ from the uniform distribution over $[0, \theta]$ consider two estimators of θ:

$$\hat{\theta}_{\text{ML}}(\{x_n\}) = \max_n (x_n)$$

$$\hat{\theta}_{\text{alt}}(\{x_n\}) = \frac{2}{N} \sum_n x_n$$







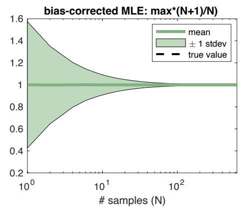
Two estimators for range of a uniform distribution

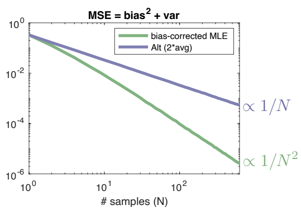
Given N samples $\{x_n\}$ from the uniform distribution over $[0, \theta]$ consider two estimators of θ:

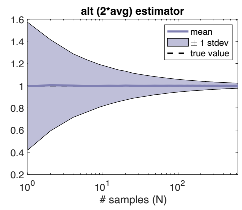
$$\hat{\theta}_{\text{cML}}(\{x_n\}) = \frac{N+1}{N} \max_n(x_n)$$

$$\hat{\theta}_{\text{alt}}(\{x_n\}) = \frac{2}{N} \sum_n x_n$$

bias-corrected







Example ML Estimators - Continuous

Uniform: $p(x|\theta) = \begin{cases} \frac{1}{\theta} & 0 \le x \le \theta \\ 0 & \text{otherwise} \end{cases}$

$$\hat{\theta}_{\text{ML}} = \max_n \{x_n\}$$

(Note: this is biased!)

$$\hat{\theta}_{\text{cML}} = \frac{N+1}{N} \hat{\theta}_{\text{ML}}$$

Gaussian: $p(x|\mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$$\hat{\mu}_{\text{ML}} = \frac{\sum_n x_n}{N}$$

$$\hat{\sigma}^2_{\text{ML}} = \frac{\sum_n (x_n - \hat{\mu})^2}{N}$$

(Note: this is biased!)

$$\hat{\sigma}^2_{\text{cML}} = \frac{N}{N-1} \hat{\sigma}^2_{\text{ML}}$$

[on board]

Properties of ML estimators

- Bias: the MLE is *asymptotically unbiased* and *Gaussian*, but can only rely on these if:
  - you have lots of data
  - the MLE can be computed
  - the likelihood model is correct
- Variance (confidence intervals / error bars):
  - S.E.M. (relevant for sample averages only)
  - second deriv of NLL (multi-D: "Hessian")
  - simulation (resample from $p(x|\hat{\theta})$)
  - bootstrapping (resample from *the data,* with replacement)

Bootstrapping

- "The Baron had fallen to the bottom of a deep lake. Just when it looked like all was lost, he thought to pick himself up by his own bootstraps"
  [Adventures of Baron von Munchausen, by Rudolph Erich Raspe]
- A **(re)sampling** method for computing estimator dispersion (incl. stdev error bars or confidence intervals)
- Idea: instead of looking at distribution of estimates across repeated experiments, look across repeated resampling (with replacement) from the *existing* data ("bootstrapped" data sets)

HEART ATTACK RISK FOUND TO BE CUT BY TAKING ASPIRIN

LIFESAVING EFFECTS SEEN

Study Finds Benefit of Tablet Every Other Day Is Much Greater Than Expected

[New York Times, 27 Jan 1987]

The summary statistics in the newspaper article are very simple:

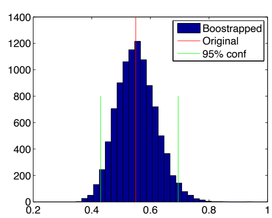
	heart attacks (fatal plus non-fatal)	subjects
aspirin group:	104	11037
placebo group:	189	11034

$$\widehat{\theta} = \frac{104/11037}{189/11034} = .55. \quad (1.1)$$

If this study can be believed, and its solid design makes it very believable, the aspirin-takers only have 55% as many heart attacks as placebo-takers.

Of course we are not really interested in $\widehat{\theta}$, the estimated ratio. What we would like to know is θ, the true ratio

Histogram of bootstrap estimates:



=> with 95% confidence,

$$0.43 < \theta < 0.7$$

[Efron & Tibshirani '98]

---

	strokes	subjects	
aspirin group:	119	11037	
placebo group:	98	11034	(1.3)

For strokes, the ratio of rates is

$$\widehat{\theta} = \frac{119/11037}{98/11034} = 1.21. \quad (1.4)$$

It now looks like taking aspirin is actually harmful. However the interval for the true stroke ratio θ turns out to be

$$.93 < \theta < 1.59 \quad (1.5)$$

with 95% confidence. This includes the neutral value $\theta = 1$, at which aspirin would be no better or worse than placebo vis-à-vis strokes. In the language of statistical hypothesis testing, aspirin was found to be significantly beneficial for preventing heart attacks, but not significantly harmful for causing strokes.

[Efron & Tibshirani '98]

Bayesian Inference

"Posterior" "Likelihood" "Prior"

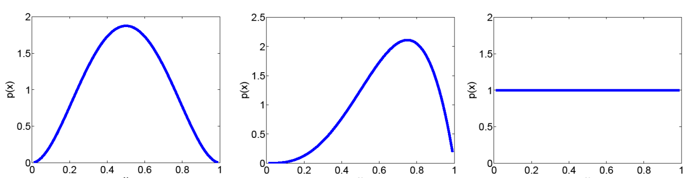
$$p(\theta \mid \text{data}) = \frac{p(\text{data} \mid \theta)\, p(\theta)}{p(\text{data})}$$

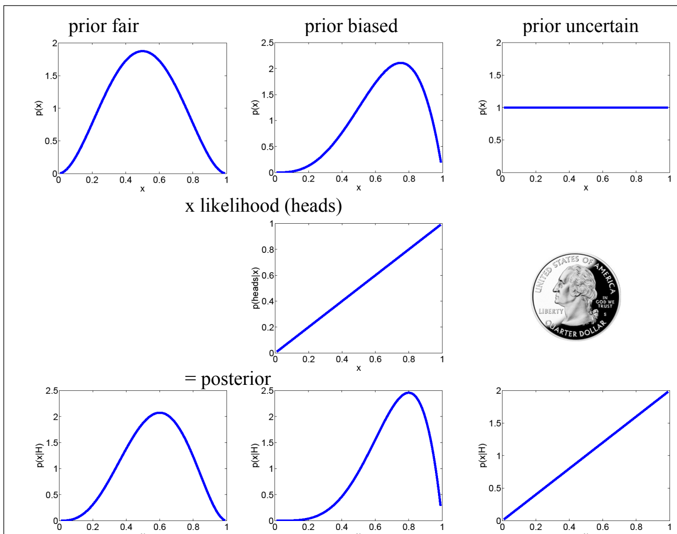
Normalization factor

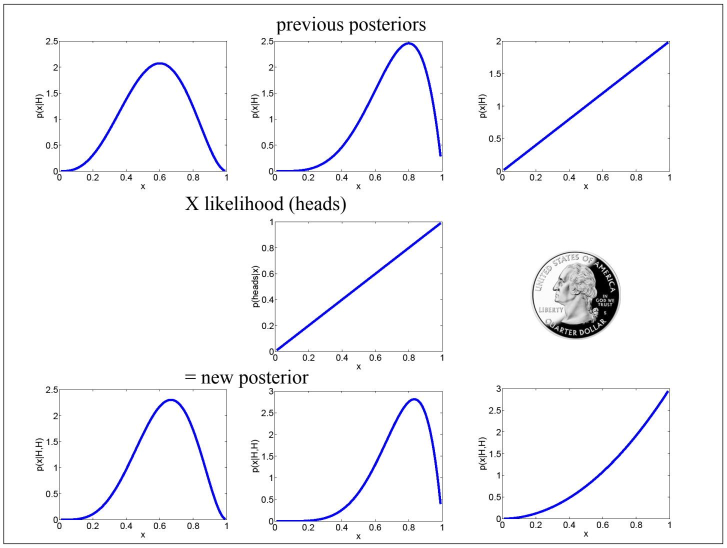
Example: Posterior for coin

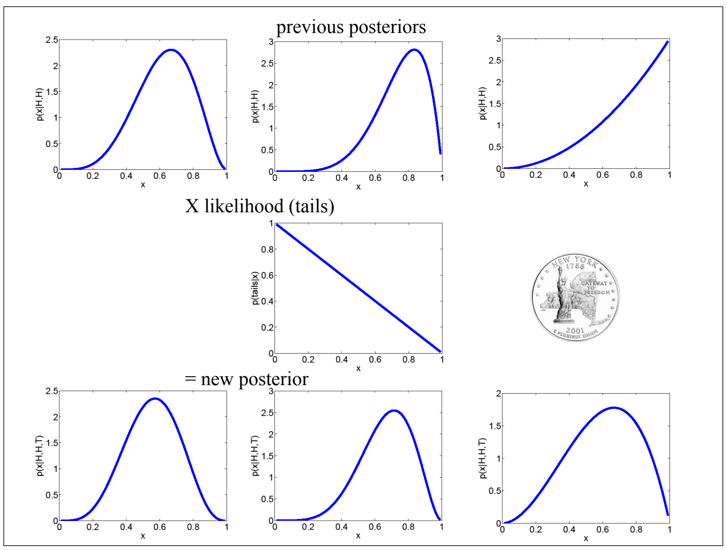
infer whether a coin is fair by flipping it repeatedly
here, x is the probability of heads (50% is fair)
$y_{1\ldots n}$ are the outcomes of flips

Consider three different priors:
suspect fair — suspect biased — no idea

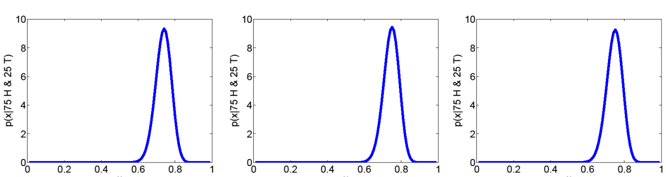








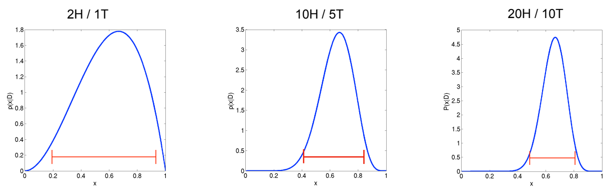
Posteriors after observing 75 heads, 25 tails



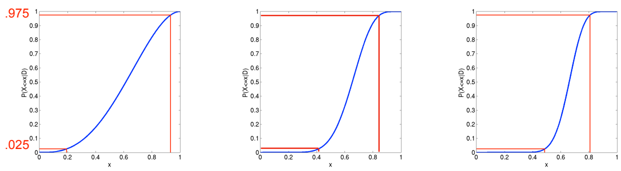
→prior differences are ultimately overwhelmed by data

Bayesian confidence intervals

PDFs



CDFs, and 95% confidence intervals



Bayesian inference: Gaussian case

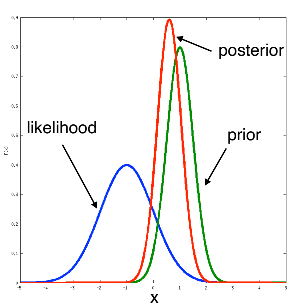
For measurements with Gaussian noise, and assuming a Gaussian prior:

- posterior is Gaussian, allowing sequential updating
- precision is sum of measurement and prior precisions
- mean is precision-weighted average of prior mean and measurement
- explains "regression to the mean", as

Bayesian inference: Gaussian case

$$y = x + n, \quad x \sim N(\mu_x, \sigma_x), \; n \sim N(0, \sigma_n)$$

$$\begin{aligned} p(x|y) &\propto p(y|x)p(x) \\ &\propto e^{-\frac{1}{2}\left[\frac{1}{\sigma_n^2}(x-y)^2\right]} e^{-\frac{1}{2}\left[\frac{1}{\sigma_x^2}(x-\mu_x)^2\right]} \\ &= e^{-\frac{1}{2}\left[\left(\frac{1}{\sigma_n^2}+\frac{1}{\sigma_x^2}\right)x^2 - 2\left(\frac{y}{\sigma_n^2}+\frac{\mu_x}{\sigma_x^2}\right)x + \ldots\right]} \end{aligned}$$



Completing the square shows that this posterior is also Gaussian, with

$$\frac{1}{\sigma^2} = \frac{1}{\sigma_n^2} + \frac{1}{\sigma_x^2}$$

$$\mu = \left(\frac{y}{\sigma_n^2} + \frac{\mu_x}{\sigma_x^2}\right) \Big/ \left(\frac{1}{\sigma_n^2} + \frac{1}{\sigma_x^2}\right)$$

The average of y and μ_x, weighted by inverse variances (a.k.a. "precisions")!

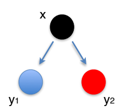
Regression to the mean

"Depressed children treated with an energy drink improve significantly over a three-month period. I made up this newspaper headline, but the fact it reports is true: if you treated a group of depressed children for some time with an energy drink, they would show a clinically significant improvement…."

"It is also the case that depressed children who spend some time standing on their head or hug a cat for twenty minutes a day will also show improvement."

- D. Kahneman

Two noisy measurements of the same variable:



$$y_1 = x + n_1 \qquad x \sim N(0, \sigma_x)$$
$$y_2 = x + n_2 \qquad n_k \sim N(0, \sigma_n), \text{ independent}$$

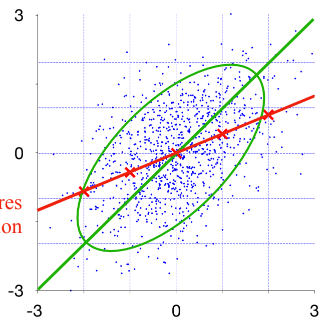
Joint measurement distribution: $\vec{y} \sim N(\vec{0}, \sigma_x^2 \vec{1}\vec{1}^T + \sigma_n^2 I)$

LS Regression:

$$\hat{\beta} = \arg\min_{\beta} ||y_2 - \beta y_1||^2$$
$$= \frac{\sigma_x^2}{\sigma_x^2 + \sigma_n^2}$$

$$\mathbb{E}(y_2 | y_1) = \hat{\beta}\, y_1$$

"regression to the mean"



The hierarchy of statistical estimators

- Maximum likelihood (ML): $\hat{x}(\vec{d}) = \arg\max_x p(\vec{d}|x)$
- Maximum a posteriori (MAP): (requires prior, $p(x)$) $\hat{x}(\vec{d}) = \arg\max_x p(x|\vec{d})$
- Bayes estimator (general): (requires loss, $L(x, \hat{x})$) $\hat{x}(\vec{d}) = \arg\min_{\hat{x}} \mathbb{E}\left(L(x, \hat{x}) \,\middle|\, \vec{d}\right)$
- Bayes least squares (BLS): (special case, squared loss) $\hat{x}(\vec{d}) = \arg\min_{\hat{x}} \mathbb{E}\left((x - \hat{x})^2 \,\middle|\, \vec{d}\right) = \mathbb{E}\left(x \,\middle|\, \vec{d}\right)$