Mathematical Tools
for Neural and Cognitive Science

Fall semester, 2023

Section 1: Linear Algebra

Linear Algebra

"Linear algebra has become as basic and as applicable as calculus, and fortunately it is easier"

- Gilbert Strang, *Linear Algebra and its Applications*

… and this is even more true today than when the book was published!
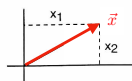

Vectors

Ordered lists of numbers, depicted in 3 ways:
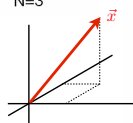
$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_N \end{pmatrix}$$

In two or three dimensions, we can draw these as arrows:

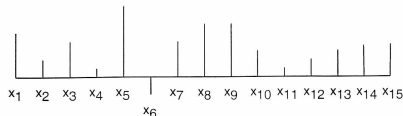

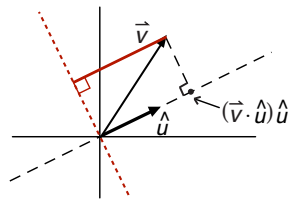In higher dimensions, we typically must resort to a "spike-plot":

Vector operations

- scalar multiplication
- addition, vector spaces
- length, unit vectors
- inner product (a.k.a. "dot" product)
  - definition/notation: sum of pairwise products
  - geometry: cosines, squared length, orthogonality test

[on board: geometry]

Inner product with a unit vector

- projection onto line
- distance to line/plane
- change of coordinates

[on board: geometry]

Vectors as "operators"

- "averager"
- "windowed averager"
- "smooth averager"
- "local differencer"
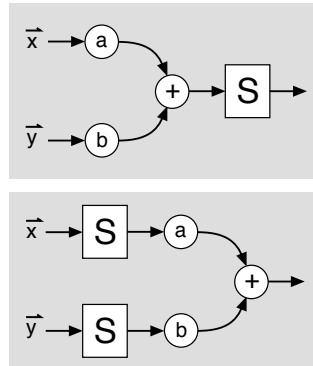- "component selector"

[on board]

Linear System

S is a linear system if (and only if) it obeys the principle of superposition:

$$S(a\vec{x} + b\vec{y}) = aS(\vec{x}) + bS(\vec{y})$$

For *any* input vectors $\{\vec{x}, \vec{y}\}$, and *any* scalars $\{a, b\}$, the two diagrams at the right must produce the same response.
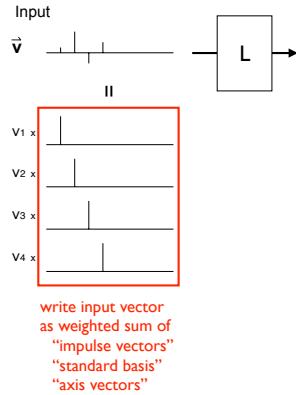
Linear Systems

- Very well understood (150+ years of effort)

- Excellent design/characterization toolbox

- An idealization (they do not exist!)

- Useful nevertheless:
  - conceptualize fundamental issues
  - provide baseline performance
  - provide building blocks for more complex models
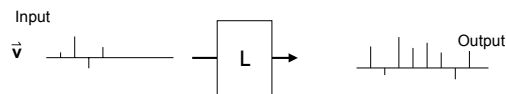
Implications of Linearity

Implications of Linearity

Input

$\vec{v}$

L

=

v_1 x

v_2 x

v_3 x

v_4 x

write input vector
as weighted sum of
"impulse vectors"
"standard basis"
"axis vectors"

Implications of Linearity

Input

$\vec{v}$

L

Output

=

v_1 x

v_2 x

v_3 x

v_4 x

L

v_1 x

v_2 x

v_3 x

v_4 x

"impulse vectors"
"axis vectors"
"standard basis"
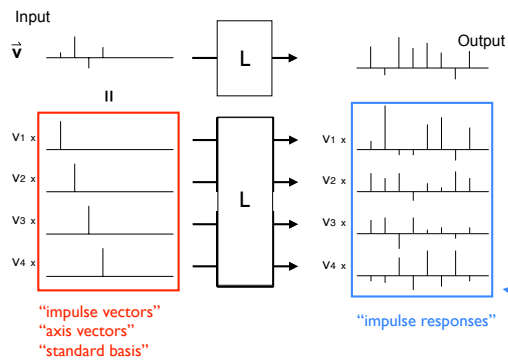
"impulse responses"

Response to *any* input can be computed from responses to impulses
This defines the operation of *matrix multiplication*

Matrix multiplication

Two interpretations of $M\vec{v}$

input perspective:
weighted sum of columns

multiply
by v_1

System response
to first axis, $\hat{e}_1$
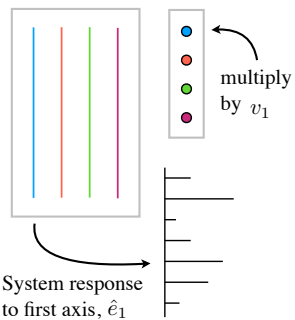
output perspective:
inner product with rows

[details on board]

Rafetto's Pasta (est. 1906)

https://raffettospasta.com

Matrix multiplication

- two interpretations of $M\vec{v}$:
  - weighted sum of columns
  - inner products with rows

- transpose A^T, symmetric matrices ($A = A^T$)

- distributive property: directly from linearity!

- associative property: cascade of two linear systems is linear. Defines matrix multiplication.

[details on board]

Cascaded linear systems => product of matrices

Matrix multiplication

- two interpretations of $M\vec{v}$:
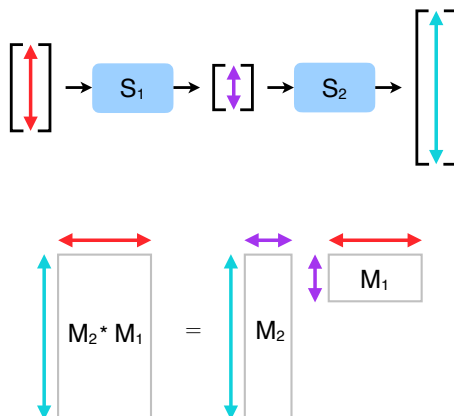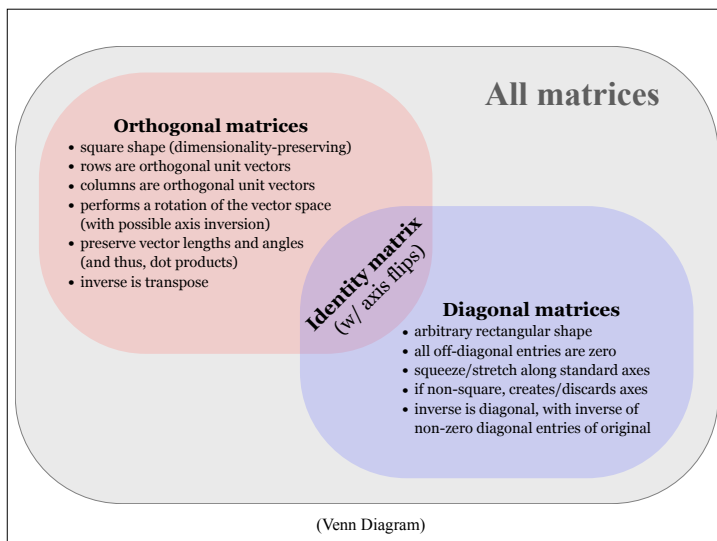 - "input perspective": weighted sum of columns
  - "output perspective": inner product with rows

- transpose A^T, symmetric matrices ($A = A^T$)

- distributive property: directly from linearity!

- associative property: cascade of two linear systems is linear. Defines matrix multiplication.

- generally *not* commutative ($AB \neq BA$), but note that $(AB)^T = B^T A^T$

- vectors as matrices: Inner products, Outer products

[details on board]

All matrices

Orthogonal matrices
- square shape (dimensionality-preserving)
- rows are orthogonal unit vectors
- columns are orthogonal unit vectors
- performs a rotation of the vector space (with possible axis inversion)
- preserve vector lengths and angles (and thus, dot products)
- inverse is transpose

Identity matrix (w/ axis flips)

Diagonal matrices
- arbitrary rectangular shape
- all off-diagonal entries are zero
- squeeze/stretch along standard axes
- if non-square, creates/discards axes
- inverse is diagonal, with inverse of non-zero diagonal entries of original

(Venn Diagram)

Singular Value Decomposition (SVD)

Any matrix M can be factorized as
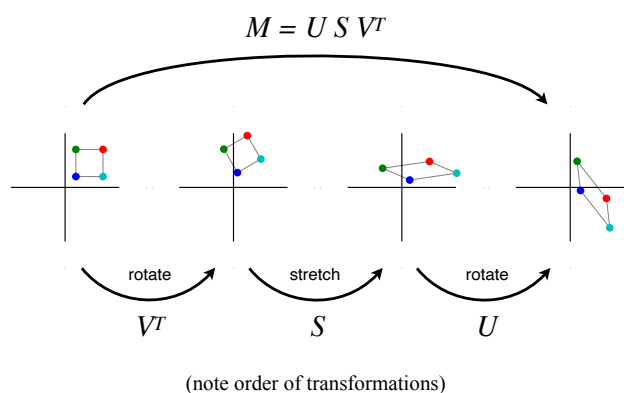
$$M = U\,S\,V^T$$

with U, V orthogonal, S diagonal

- geometry: "rotate, stretch, rotate"
- columns of V are basis for *input* coordinate system
- columns of U are basis for *output* coordinate system
- S rescales axes, and determines what "gets through"

[details on board]

SVD geometry (in 2D)

Apply *M* to four vectors (heads at colored points):

$$M = U\, S\, V^T$$

rotate   stretch   rotate

V^T S U

(note order of transformations)

Singular Value Decomposition (SVD)

> *Any* matrix *M* can be factorized as
> $$M = U\, S\, V^T$$
> with *U, V* orthogonal, *S* diagonal

- unique, up to permutations and sign flips
- sum of "outer products"
- nullspace and rangespace
- inverse and pseudo-inverse

[details on board]

$$M\vec{x} = \sum_k \hat{u}_k \left(s_k \left(\hat{v}_k^T \vec{x} \right) \right) = \sum_k s_k \left(\hat{u}_k \hat{v}_k^T \right) \vec{x} \quad \text{(sum of outer products)}$$

"singular values"

M U S V^T

s_1 s_2 s_3

(all zeros)

orthogonal basis for input space

orthogonal basis for output space

M U S V^T

"singular values"

s_1 s_2 s_3

(all zeros)

orthogonal basis for input space

orthogonal basis for "range space"

$=$

M U S V^T

s_1
s_2
0

(all zeros)

orthogonal basis for "null space"

orthogonal basis for "range space"

$=$

M U S V^T

s_1
0
0

(all zeros)

orthogonal basis for "null space"

orthogonal basis for "range space"

$=$