Mathematical Tools
for Neural and Cognitive Science

Fall semester, 2022

Section 1: Linear Algebra

Linear Algebra

"Linear algebra has become as basic and as applicable as calculus, and fortunately it is easier"

- Gilbert Strang, *Linear Algebra and its Applications*

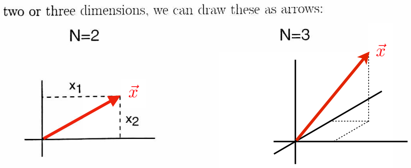
… and this is even more true today than when the book was published!

Vectors

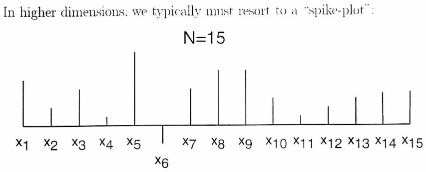
Ordered lists of numbers, depicted in 3 ways:

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_N \end{pmatrix}$$

In two or three dimensions, we can draw these as arrows:



In higher dimensions, we typically must resort to a "spike-plot":

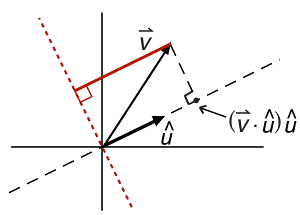


Vector operations

- scalar multiplication
- addition, vector spaces
- length, unit vectors
- inner product (a.k.a. "dot" product)
  - definition/notation: sum of pairwise products
  - geometry: cosines, squared length, orthogonality test

[on board: geometry]

Inner product with a unit vector



- projection onto line
- distance to line/plane
- change of coordinates

[on board: geometry]

Vectors as "operators"

- "averager"
- "windowed averager"
- "smooth averager"
- "local differencer"
- "component selector"

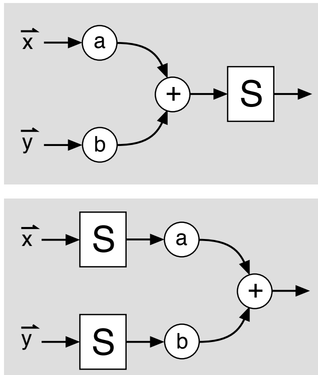
[on board]

Linear System

S is a linear system if (and only if) it obeys the principle of superposition:

$$S(a\vec{x} + b\vec{y}) = aS(\vec{x}) + bS(\vec{y})$$

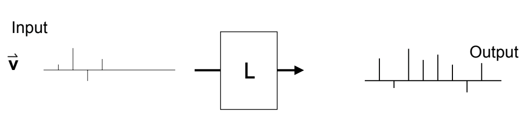
For *any* input vectors $\{\vec{x}, \vec{y}\}$, and *any* scalars $\{a, b\}$, the two diagrams at the right must produce the same response.



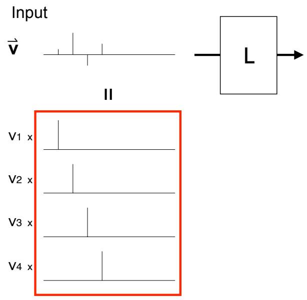
Linear Systems

- Very well understood (150+ years of effort)
- Excellent design/characterization toolbox
- An idealization (they do not exist!)
- Useful nevertheless:
  - conceptualize fundamental issues
  - provide baseline performance
  - provide building blocks for more complex models

Implications of Linearity

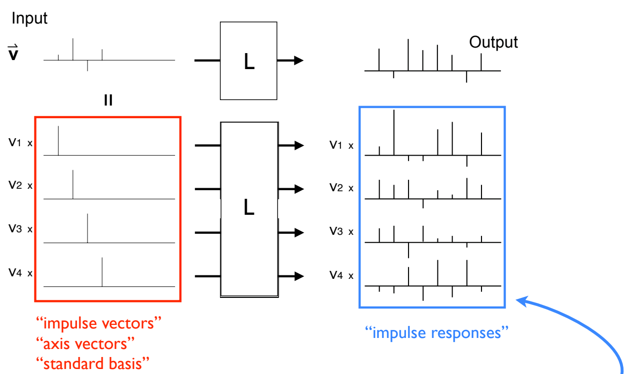


Implications of Linearity



write input vector
as weighted sum of
"impulse vectors"
"standard basis"
"axis vectors"

Implications of Linearity

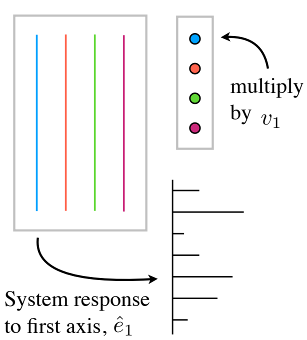


Response to *any* input can be computed from responses to impulses
This defines the operation of *matrix multiplication*

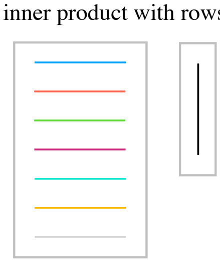
Matrix multiplication

Two interpretations of $M\vec{v}$

input perspective:
weighted sum of columns



output perspective:
inner product with rows



[details on board]

Rafetto’s Pasta (est. 1906)

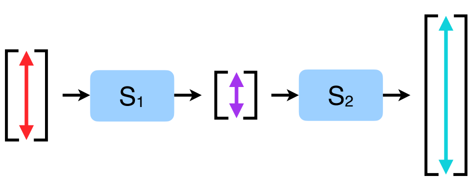
https://raffettospasta.com

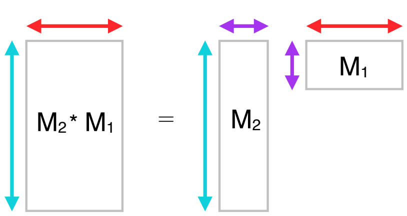
Matrix multiplication

- two interpretations of $M\vec{v}$:
  - weighted sum of columns
  - inner products with rows
- transpose A^T, symmetric matrices ($A = A^T$)
- distributive property: directly from linearity!
- associative property: cascade of two linear systems is linear. Defines matrix multiplication.

[details on board]

Cascaded linear systems => product of matrices

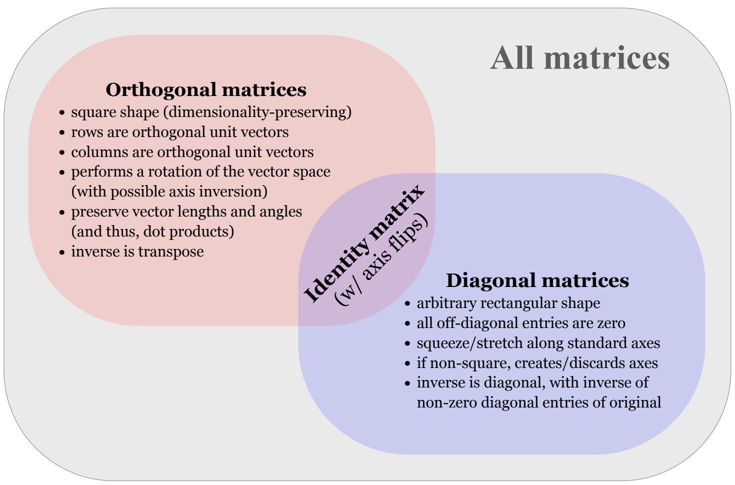




Matrix multiplication

- two interpretations of $M\vec{v}$:
  - "input perspective": weighted sum of columns
  - "output perspective": inner product with rows
- transpose A^T, symmetric matrices ($A = A^T$)
- distributive property: directly from linearity!
- associative property: cascade of two linear systems is linear. Defines matrix multiplication.
- generally *not* commutative ($AB \neq BA$), but note that $(AB)^T = B^T A^T$
- vectors as matrices: Inner products, Outer products

[details on board]



(Venn Diagram)

Singular Value Decomposition (SVD)

Any matrix M can be factorized as

$$M = U\, S\, V^T$$

with U, V orthogonal, S diagonal

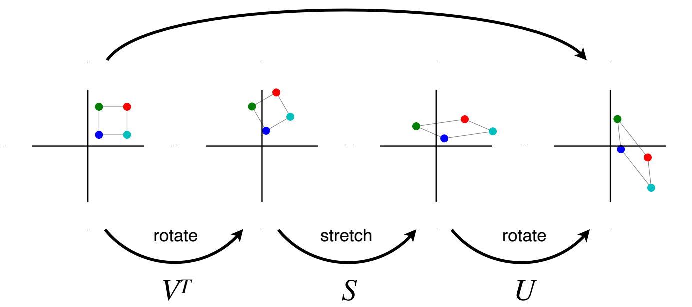
- geometry: "rotate, stretch, rotate"
- columns of V are basis for *input* coordinate system
- columns of U are basis for *output* coordinate system
- S rescales axes, and determines what "gets through"

[details on board]

SVD geometry (in 2D)

Apply M to four vectors (heads at colored points):

$$M = U\, S\, V^T$$



(note order of transformations)

Singular Value Decomposition (SVD)

Any matrix M can be factorized as

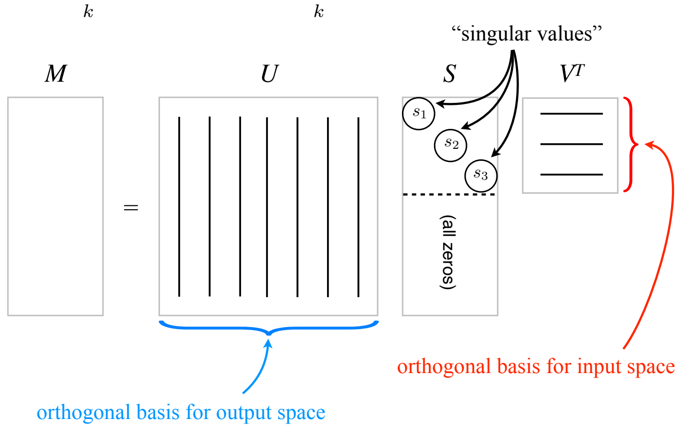
$$M = U\, S\, V^T$$

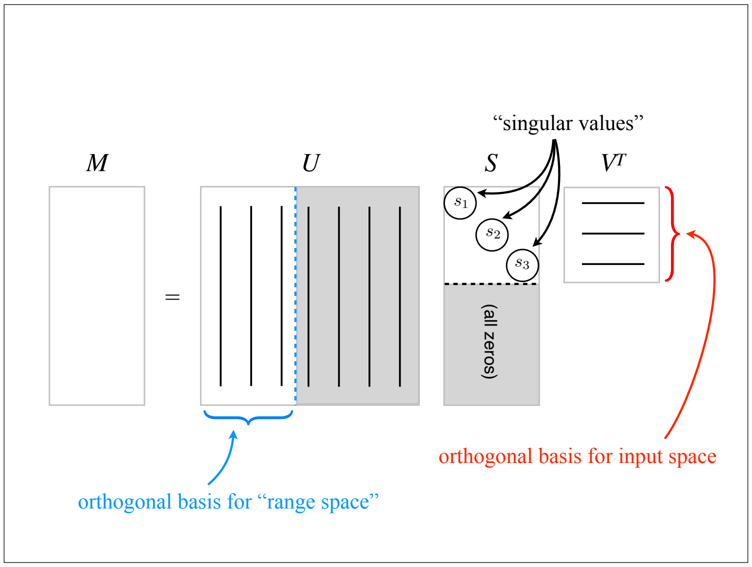
with U, V orthogonal, S diagonal

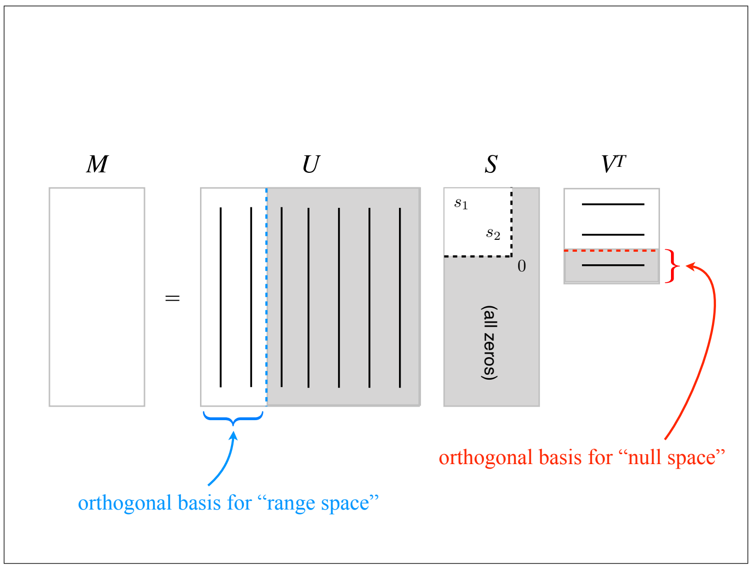
- unique, up to permutations and sign flips
- sum of "outer products"
- nullspace and rangespace
- inverse and pseudo-inverse

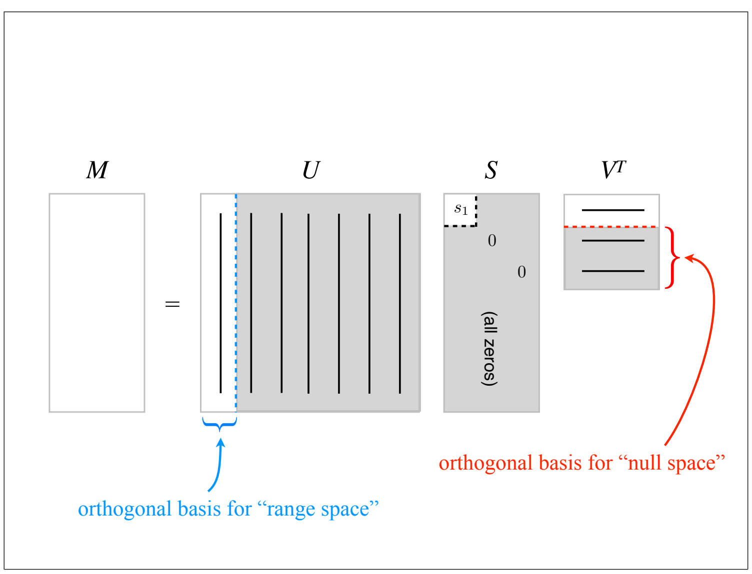
[details on board]

$$M\vec{x} = \sum_k \hat{u}_k \left(s_k \left(\hat{v}_k^T \vec{x}\right)\right) = \sum_k s_k \left(\hat{u}_k \hat{v}_k^T\right) \vec{x}$$ (sum of outer products)




"singular values"
M
U
S
V^T
s_1
s_2
s_3
=
(all zeros)
orthogonal basis for input space
orthogonal basis for "range space"


M
U
S
V^T
s_1
s_2
0
=
(all zeros)
orthogonal basis for "null space"
orthogonal basis for "range space"


M
U
S
V^T
s_1
0
0
=
(all zeros)
orthogonal basis for "null space"
orthogonal basis for "range space"