PSYCH-GA.2211/NEURL-GA.2201 – Fall 2020
Mathematical Tools for Neural and Cognitive Science

Homework 3

Due: 27 Oct 2020
(late homeworks penalized 10% per day)

See the course web site for submission details. Reminder: rather than using the functions pinv() and norm(), use the linear algebra tools we learned in class. Do yourself a favor, and don't wait until the day before the due date... *start now*!

1. **LSI system characterization.** You are trying to experimentally characterize three auditory neurons, in terms of their responses to sounds. For purposes of this problem, the responses of these neurons are embodied in compiled matlab functions `unknownSystemX.p`, (with X=1, 2, 3), each of which takes an input column vector of length $N = 64$ whose elements represent sound pressure over time. The response of each is a column vector (of the same length) representing the mean spike count over time. Your task is to examine them to see if they: i) behave like they are linear; and/or ii) shift-invariant; with/without iii) circular (i.e. periodic) boundary handling . For each neuron,

   (a) "Kick the tires" by measuring the response to an impulse in the first position of an input vector. Check that this impulse response is shift-invariant by comparing to the response to an impulse at positions $n = 2, 4, 8$. Use different n to determine how the system handles inputs near the boundary. Also check that the response to a sum of any two of these impulses is equal to the sum of their individual responses. Be sure to describe your findings.

   (b) If the previous tests succeeded, examine the response of the system to sinusoids with frequencies $\{2\pi/N, 4\pi/N, 8\pi/N, 16\pi/N\}$, and random phases, and check whether the outputs are sinusoids of the same frequency (i.e., verify that the output vector lies completely in the subspace containing all the sinusoids of that frequency). [note: Make the input stimuli positive, by adding one to each sinusoid, and the responses should then be positive (mean spike counts)].

   (c) If the previous tests succeeded, verify that the change in amplitude and phase from input to output is predicted by the amplitude (`abs`) and phase (`angle`) of the corresponding terms of the Fourier transform of the impulse response. If not, explain which property (linearity, or shift-invariance, or both) seems to be missing from the system. If so, do you think that the combination of all tests *guarantees* that the system is linear and shift-invariant? What combination of tests would provide such a guarantee?

2. **Neuron in visual cortex.** The response properties of neurons in primary visual cortex (area V1) are often described using linear filters. We'll examine a one-dimensional cross-section of the most common choice, known as a "Gabor filter" (named after Electrical Engineer/Physicist Denis Gabor, who developed it in 1946 for use in signal processing).

   (a) Create a one-dimensional linear filter that is a product of a Gaussian and a sinusoid, $\exp\left(-\frac{n^2}{2\sigma^2}\right)\cos(\omega n)$, with parameters $\sigma = 3.5$ and $\omega = 2\pi * 10/64$ samples. The filter should contain 31 samples, and the Gaussian should be centered on the middle (16th) sample. Plot the filter to verify that it looks like what you'd expect. Plot the amplitude of the Fourier transform of this filter, sampled at 64 locations (MATLAB's `fft` function takes an optional additional argument). What kind of filter is this? Why does it have this shape, and how is the shape related to the choice of parameters (σ, ω)? Specifically, how does the Fourier amplitude change if you alter each of these parameters?

(b) If you were to convolve this filter with sinusoids of different frequencies, which of them would produce a response with the largest amplitude? Obtain this answer by reasoning about the equation defining the filter (above), and also by finding the maximum of the computed Fourier amplitudes (using the `max` function), and verify that the answers are the same. Compute the *period* of this sinusoid, measured in units of sample spacing (hint: this is the inverse of its frequency, in cycles/sample), and verify by eye that this is roughly matched to the oscillations in the graph of the filter itself. Which two sinusoids would produce responses with about 25% of this maximal amplitude?

(c) Create three unit-amplitude 64-sample sinusoidal signals at the three frequencies (low, mid, high) that you found in part (b). Convolve the filter with each, and verify that the amplitude of the response is approximately consistent with the answers you gave in part (b). (hint: to estimate amplitude, you'll either need to project the response onto sine and cosine of the appropriate frequency, or compute the DFT of the response and measure the ampitude at the appropriate frequency).

(d) Verify the convolution theorem. Apply the Fourier transform to each of your three stimuli. Multiply each by the Fourier transform of the Gabor filter. Inverse Fourier transform the results and verify that the imaginary part is zero, and the real part is equal to the result you obtain from the convolution.

3. **Deconvolution of the Haemodynamic Response.** Neuronal activity causes local changes in deoxyhemoglobin concentration in the blood, which can be measured using magnetic resonance imaging (MRI). One drawback of this is that the haemodynamic response is both delayed and slower than the underlying neural responses. We can model the delay and spread of the measurements relative to the neural signals using a linear shift-invariant system:

$$r(n) = \sum_k x(n-k)h(k), \tag{1}$$

where $x(n)$ is an input signal delivered over time (for example, a sequence of light intensities), $h(k)$ is the haemodynamic response to a single light flash at time $k = 0$ (i.e., the impulse response of the MRI measurement), and $r(n)$ is the MRI response to the full input signal.

In the file `hrfDeconv.mat`, you will find a response vector r and an input vector x containing a sequence of impulses (indicating flashes of light). Your goal is to estimate the HRF, h, from the data. Each of these signals are sampled at 1 Hz. Plot vectors r and x versus time to get a sense for the data. (Use the `stem` command for x, and label the x-axis).

(a) Convolution is linear, and thus we can re-write the equation above as a matrix multiplication, $r = Xh$, where h is a vector of length M, and X is an $N + M - 1 \times M$ matrix (N is the length of the input x). Write a matlab function `createConvMat`, that takes as arguments an input vector x and M (the dimensionality of h) and generates a matrix X such that the response $r = Xh$ is as defined in Eq. (1) for any h. Verify that the matrix generated by your function produces the same response as MatLab's conv function when applied to a few random h vectors of length $M = 15$. Visualize the matrix X as an image (evaluate imagesc(X)), and describe its structure.

(b) Now, given the X generated by your function for $M = 15$, solve for h by formulating a least-squares regression problem:

$$h_{\text{opt}} = \arg\min_h ||r - Xh||^2$$

Plot h_{opt} as a function of time (label your x-axis, including units). How would you describe it? How long does it last?

(c) It's often easier to understand an LSI system by viewing it in the frequency domain. Plot the power-spectrum of the HRF (i.e. $|\mathcal{F}(h)|^2$, where $\mathcal{F}(h)$ is the Fourier transform of the HRF). Plot this with the zero frequency (DC) in the middle, and label the x axis, in Hz. Based on this plot, what kind of filter is the HRF? Specifically, which frequencies does it allow to pass, and which does it block?

4. **Sampling and aliasing.** Load the file myMeasurements.mat into matlab. It contains a vector, sig, containing voltage values measured from an EEG electrode, sampled at 120Hz. Plot sig as a function of vector time (time, in seconds), using the flag 'ko-' in matlab's plot command so you can see the samples.

(a) This signal is only a small portion of the full data, and you don't want to store all those values. Create a subsampled version of the signal, which contains every *fourth* value. Plot this, against the corresponding entries of the time vector, on top of the original data (use matlab's hold function, and plot with flag 'r*-'). How does this reduced version of the data look, compared to the original? Does it provide a good summary of the original measurements? Is the subsampling operation linear? Shift-invariant? Explain.

(b) Examine your EEG result in the frequency domain. First plot the magnitude (amplitude) of the Fourier transform of the original signal, over the range $[-N/2, (N/2) - 1]$ (use fftshift). By convention, the "Delta" band corresponds to frequencies less than 4Hz, "Theta" band is 4-7Hz, "Alpha" band 8-15Hz, and "Beta" is 16-31Hz. For these data, which band shows the strongest signal? Is there any power in frequencies outside of these known bands, and if so can you explain the origin of this part of the signal?

(c) Write a function signalPart = bandWiseReconstruct(bandName) that reconstructs the signal (and plots the reconstruction) using only sinusoids from the band corresponding to the string bandName (i.e. for bandName = 'Delta' the reconstruction should be a sum of sinusoids with frequencies from 0-4Hz). Hint: to reconstruct the signal start with the Fourier transform of the signal, set the appropriate Fourier coefficients to zero, then return to the time-domain with the inverse Fourier transform.

(d) Plot the Fourier magnitude for signals downsampled by factors of 2, 3, and 4, after upsampling them back to full size (i.e., make a full-size signal filled with zeros, and set every kth sample equal to one of the subsampled values, for subsampling by factor k). What is the relationship between these plots and the original frequency plot. What has happened to the frequency components of the original signal? Does the strongest signal band change?