Mathematical Tools
for Neural and Cognitive Science

Fall semester, 2020

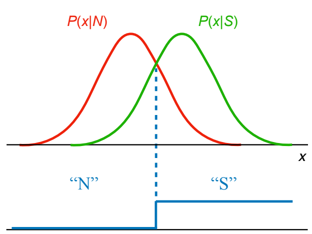
Section 5a:

Statistical Decision Theory
+
Signal Detection Theory

(357,221) = 328

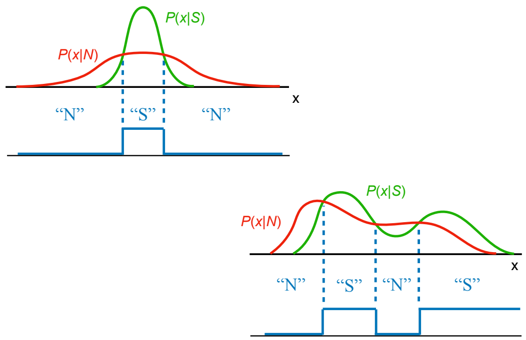
Tumor, or not?

Signal Detection Theory (binary estimation)

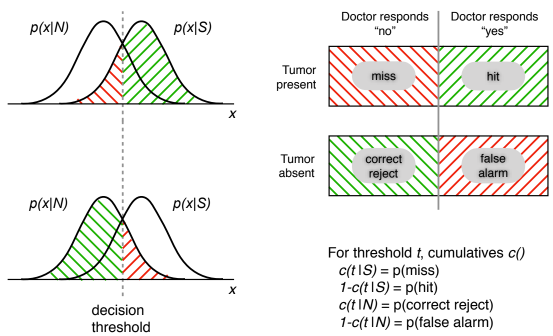


For equal-shape, unimodal, symmetric distributions,
the ML decision rule is a *threshold* function.

More generally, decision rule can have multiple thresholds…



Signal Detection Theory: Potential outcomes

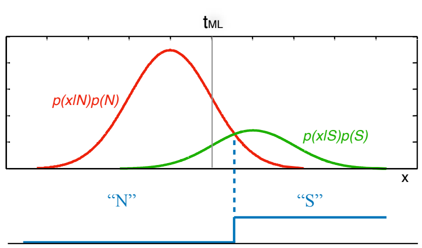


For threshold *t*, cumulatives *c()*

$c(t\,|\,S)$ = p(miss)
$1\text{-}c(t\,|\,S)$ = p(hit)
$c(t\,|\,N)$ = p(correct reject)
$1\text{-}c(t\,|\,N)$ = p(false alarm)

MAP decision rule?

MAP solution maximizes proportion of correct answers, *taking prior probability into account*.



Compared to ML threshold, the MAP threshold moves *away* from higher-probability option.

Bayes decision rule?

Incorporate values for the four possible outcomes:

Payoff Matrix

		Response: Yes	Response: No
Stimulus	S+N	V_{S+N}^{Yes}	V_{S+N}^{No}
Stimulus	N	V_{N}^{Yes}	V_{N}^{No}

Bayes Optimal Criterion

		Response: Yes	Response: No
Stimulus	S+N	V_{S+N}^{Yes}	V_{S+N}^{No}
Stimulus	N	V_{N}^{Yes}	V_{N}^{No}

$$\mathbb{E}(Yes\,|\,x) = V_{S+N}^{Yes}\,p(S+N\,|\,x) + V_{N}^{Yes}\,p(N\,|\,x)$$
$$\mathbb{E}(No\,|\,x) = V_{S+N}^{No}\,p(S+N\,|\,x) + V_{N}^{No}\,p(N\,|\,x)$$

Say yes if $\mathbb{E}(Yes\,|\,x) \geq \mathbb{E}(No\,|\,x)$

Optimal Criterion

$$\mathbb{E}(Yes\,|\,x) = V_{S+N}^{Yes}\,p(S+N\,|\,x) + V_{N}^{Yes}\,p(N\,|\,x)$$
$$\mathbb{E}(No\,|\,x) = V_{S+N}^{No}\,p(S+N\,|\,x) + V_{N}^{No}\,p(N\,|\,x)$$

Say yes if $\mathbb{E}(Yes\,|\,x) \geq \mathbb{E}(No\,|\,x)$

Say yes if $\dfrac{p(S+N\,|\,x)}{p(N\,|\,x)} \geq \dfrac{V_N^{No} - V_N^{Yes}}{V_{S+N}^{Yes} - V_{S+N}^{No}} = \dfrac{V(\text{Correct}\,|\,N)}{V(\text{Correct}\,|\,S+N)}$

Posterior odds

Apply Bayes' Rule

Posterior Likelihood Prior

$$p(S+N \mid x) = \frac{p(x \mid S+N)\,p(S+N)}{p(x)}$$

Nuisance normalizing term

$$p(N \mid x) = \frac{p(x \mid N)\,p(N)}{p(x)}, \text{ hence}$$

$$\frac{p(S+N \mid x)}{p(N \mid x)} = \left(\frac{p(x \mid S+N)}{p(x \mid N)}\right)\left(\frac{p(S+N)}{p(N)}\right)$$

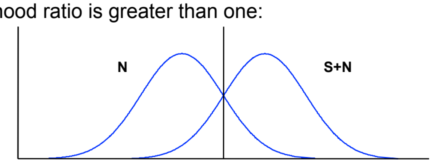
Posterior odds Likelihood ratio Prior odds

Optimal Criterion

Say yes if $\dfrac{p(S+N \mid x)}{p(N \mid x)} \geq \dfrac{V(\text{Correct} \mid N)}{\text{V}(\text{Correct} \mid S+N)}$

i.e., if $\dfrac{p(x \mid S+N)}{p(x \mid N)} \geq \dfrac{p(N)}{p(S+N)} \dfrac{V(\text{Correct} \mid N)}{\text{V}(\text{Correct} \mid S+N)} = \beta$

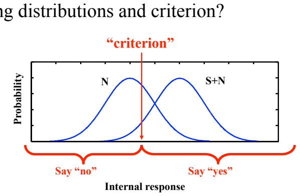
Example, if equal priors and equal payoffs, say yes if the likelihood ratio is greater than one:



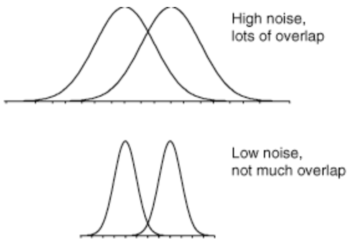
Example applications of SDT

- Vision
  - Detection (something vs. nothing)
  - Discrimination (lower vs greater level of: intensity, contrast, depth, slant, size, frequency, loudness, ...
- Memory (internal response = trace strength = familiarity)
- Neurometric function/discrimination by neurons (internal response = spike count)

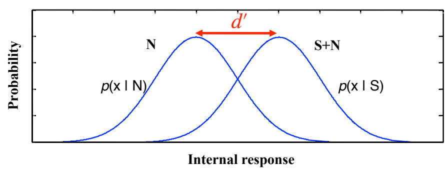
From experimental measurements, assuming human is optimal, can we determine the underlying distributions and criterion?



Signal Detection Theory: discriminability (d')



Internal response: probability of occurrence curves

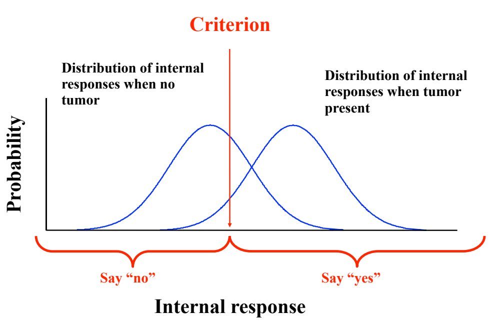


$$d' = \frac{\text{“separation”}}{\text{“width”}}$$

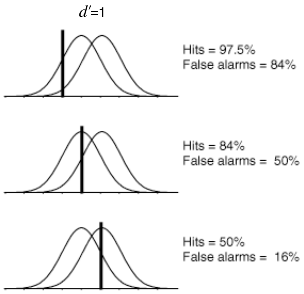
Discriminability (“d-prime”) is the normalized separation between the two distributions

Error rate is a function of d'

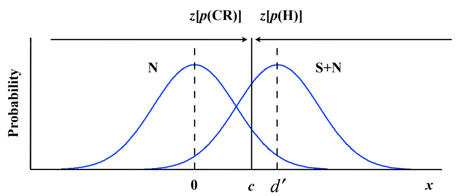
Criterion



Signal Detection Theory: Criterion



SDT: Gaussian case

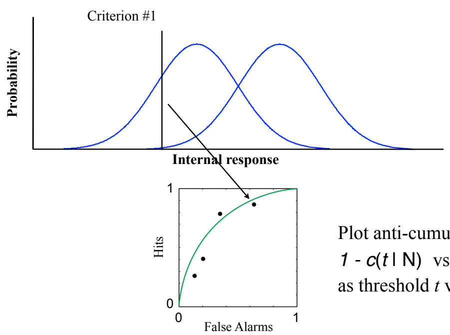


$$d' = z[p(H)] + z[p(CR)] = z[p(H)] - z[p(FA)]$$

$$\mathrm{c} = z[p(CR)] \qquad G(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

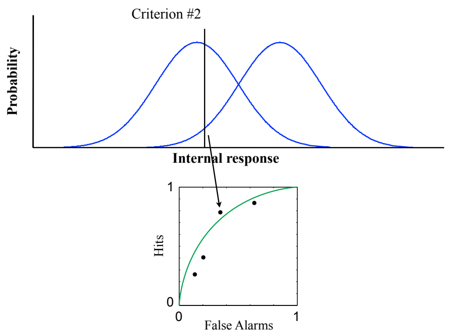
$$\beta = \frac{p(x = c \mid S+N)}{p(x = c \mid N)} = \frac{e^{-(c-d')^2/2}}{e^{-c^2/2}} \qquad (\text{Fix } \sigma = 1)$$

ROC (Receiver Operating Characteristic)

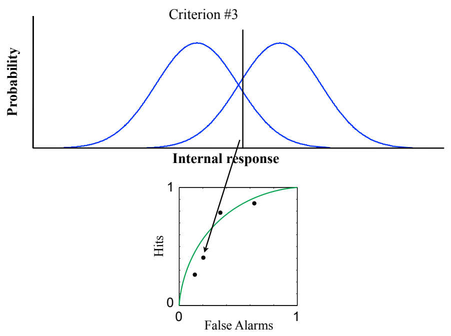


Plot anti-cumulatives:
$1 - c(t \mid \mathrm{N})$ vs. $1 - c(t \mid S)$
as threshold t varies

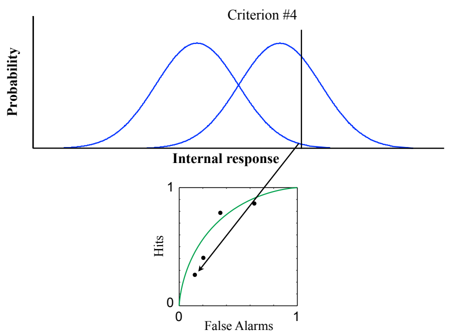
ROC (Receiver Operating Characteristic)



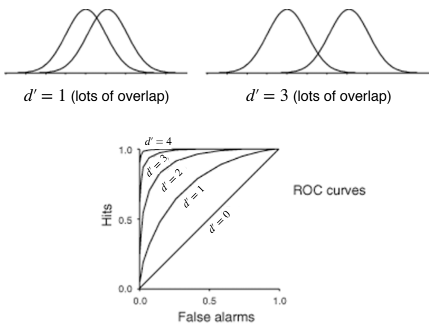
ROC (Receiver Operating Characteristic)



ROC (Receiver Operating Characteristic)



ROC (Receiver Operating Characteristic)



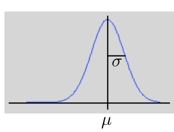
[on board: Area under curve = %correct in a 2AFC task]

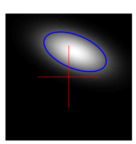
Decision/classification in multiple dimensions

- Data-driven:
  - Fisher Linear Discriminant (FLD) - maximize d'
  - Support Vector Machine (SVM) - maximize margin
- Statistical:
  - ML/MAP/Bayes under a probabilistic model
  - e.g.: Gaussian, equal covariance (same as FLD)
  - e.g.: Gaussian, unequal covariance (QDA)
- Examples:
  - Visual gender identification
  - Neural population decoding

Multi-D Gaussian densities

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



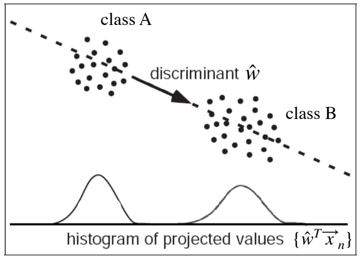


mean: [0.2, 0.8]
cov: [1.0 -0.3;
-0.3 0.4]

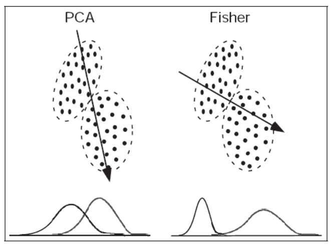
$$p(\vec{x}) = \frac{1}{\sqrt{(2\pi)^N |C|}}\, e^{-(\vec{x}-\vec{\mu})^T C^{-1} (\vec{x}-\vec{\mu})/2}$$

Linear Classifier

Find unit vector $\hat{w}$ ("discriminant") that best separates two distributions



Fisher Linear Discriminant



$$\hat{w} = C^{-1}(\vec{u}_A - \vec{u}_B)$$

$$\max_{\hat{w}} \frac{\left[\hat{w}^T(\vec{u}_A - \vec{u}_B)\right]^2}{\left[\hat{w}^T C_A \hat{w} + \hat{w}^T C_B \hat{w}\right]}$$

(note: this is "d-prime" !)

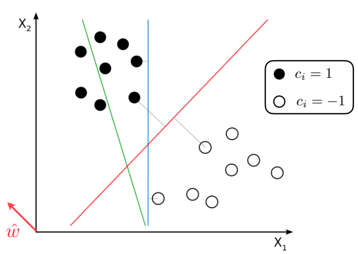
optimum: $\hat{w} = C^{-1}(\vec{u}_A - \vec{u}_B)$, where $C = \frac{1}{2}(C_A + C_B)$

Support Vector Machine (SVM)

(widely used in machine learning, has no closed form)

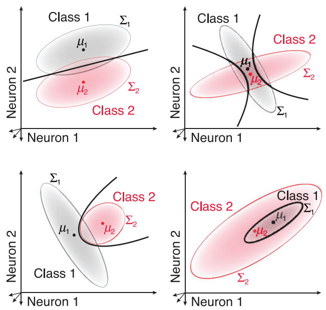
Maximize the "margin" (gap between data sets)

find largest m, and $\{\hat{w}, b\}$ s.t. $c_i(\hat{w}^T \vec{x}_i - b) \geq m, \quad \forall\, i$



ML (or MAP) classifier assuming Gaussians

Decision boundary is *quadratic*, with four possible geometries:



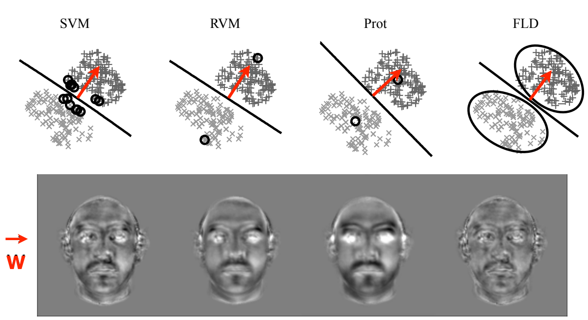
[figure: Pagan et al. 2016]

Perceptual example: Gender identification

- 200 face images (100 male, 100 female)
- Adjusted for position, size, intensity/contrast
- Labeled by 27 human subjects

[Graf & Wichmann, NIPS*03]

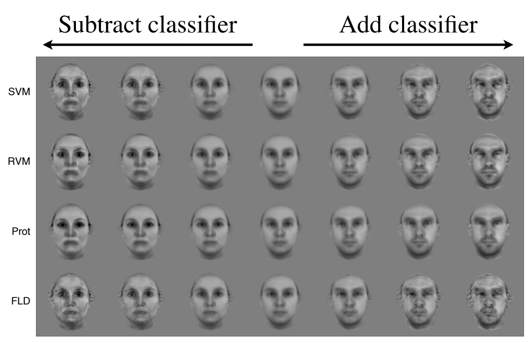
Linear classifiers



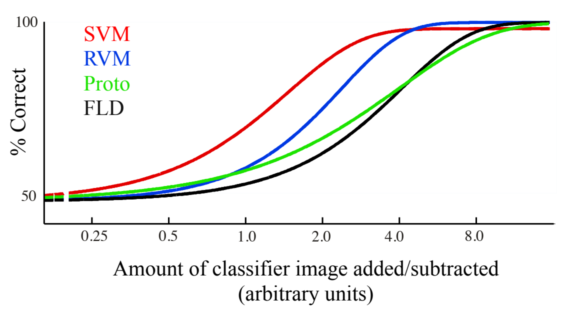
Four linear classifiers trained on subject data

Model validation/testing

- Cross-validation: Subject responses [% correct, reaction time, confidence] are explained
  - very well by SVM
  - moderately well by RVM / FLD
  - not so well by Prot
- Curse of dimensionality strongly limits this result. A more direct test: Synthesize optimally discriminable faces...



[Wichmann, et. al; NIPS*04]



[Wichmann, et. al; NIPS*04]