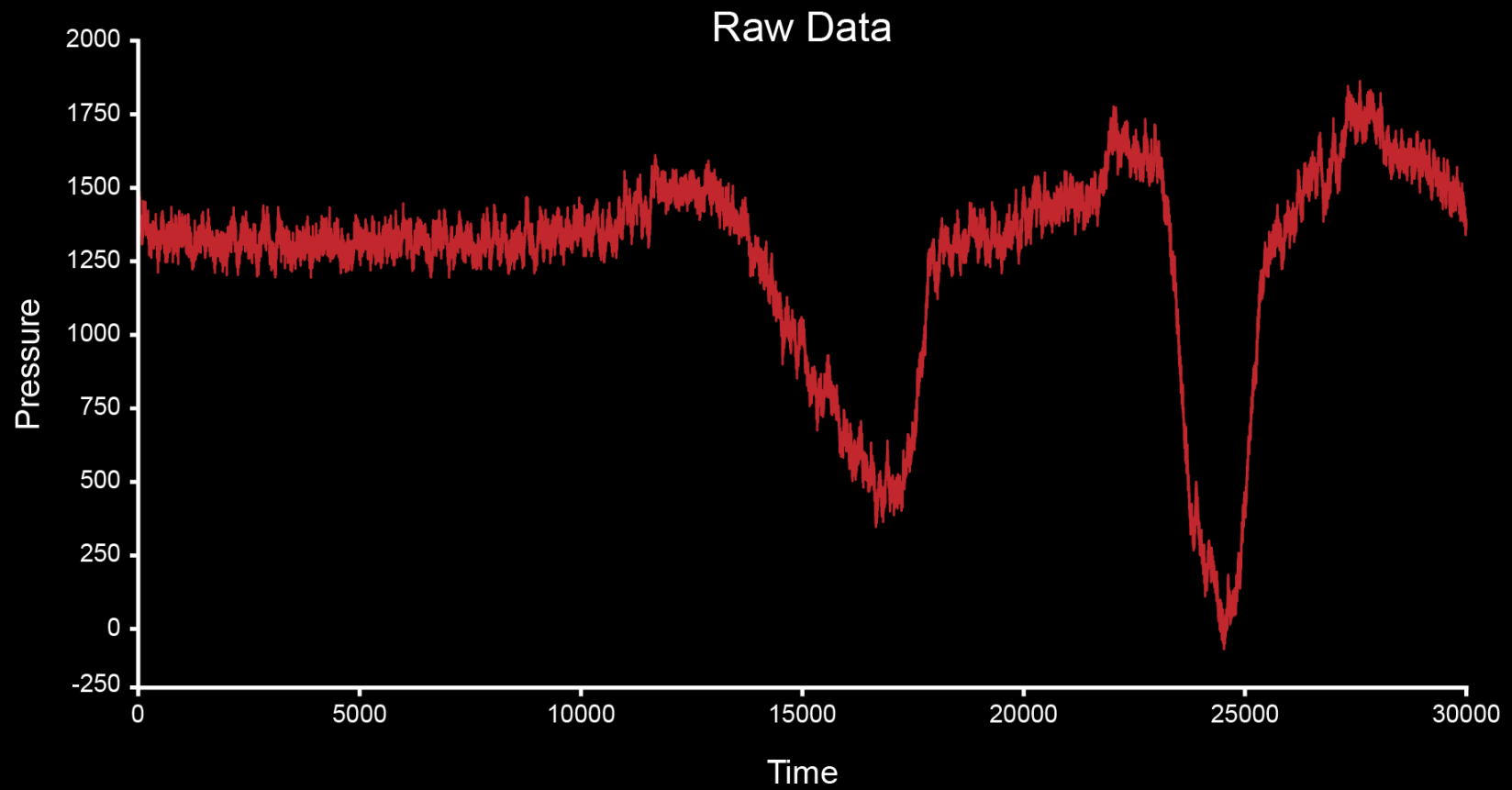
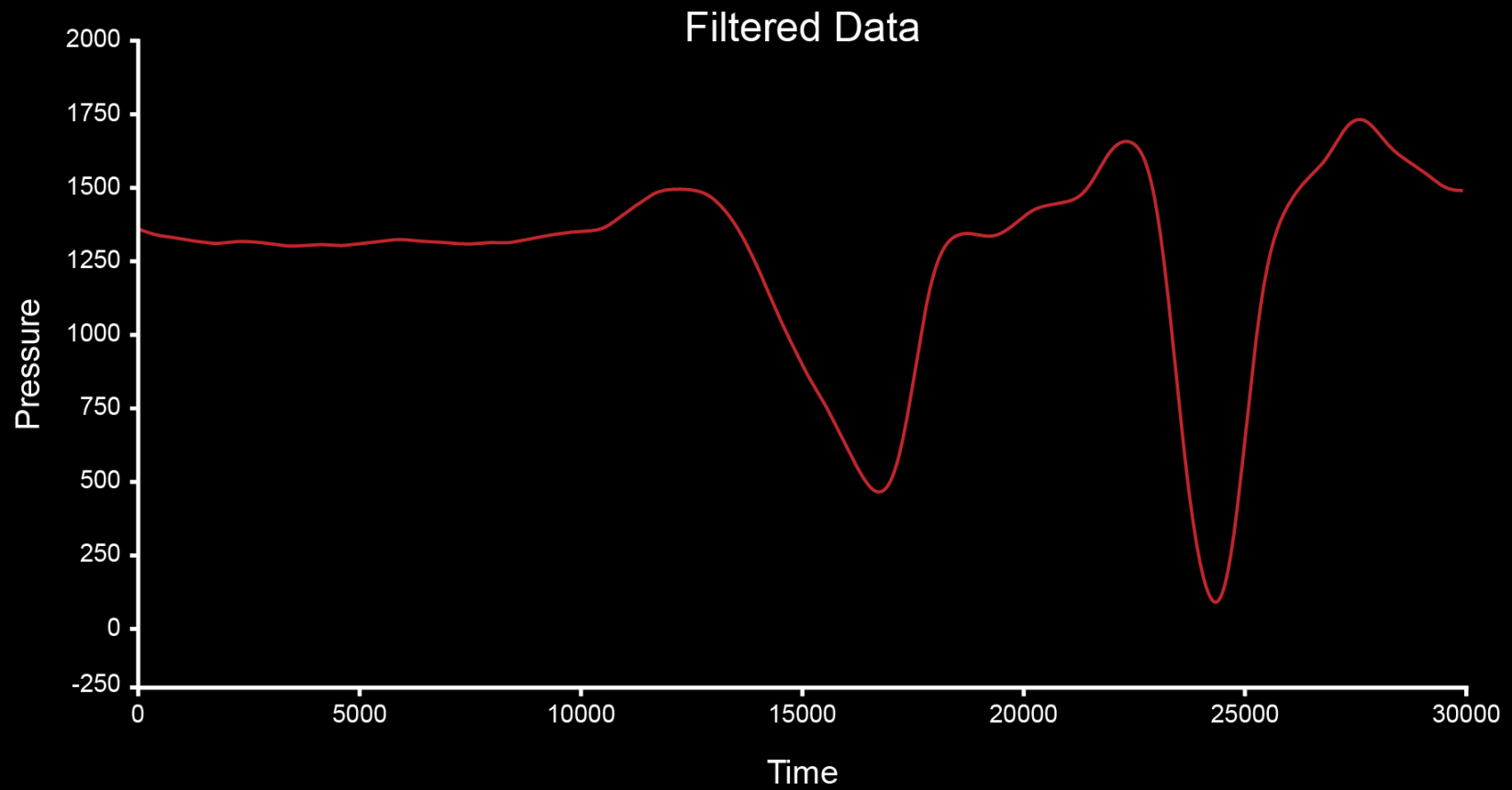


Lab VI: Fourier Transform

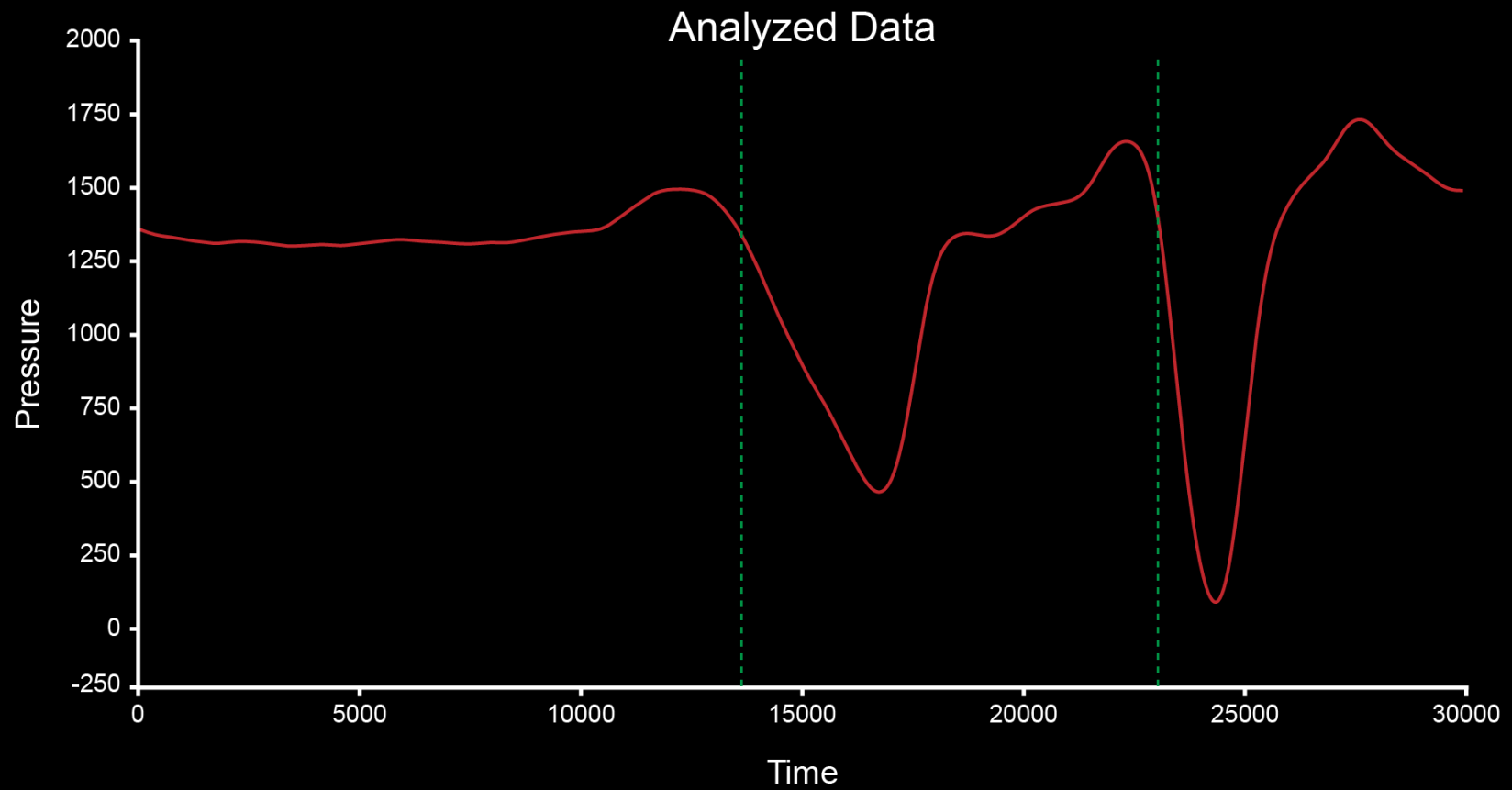
Motivating Example: removing noise



Motivating Example: removing noise

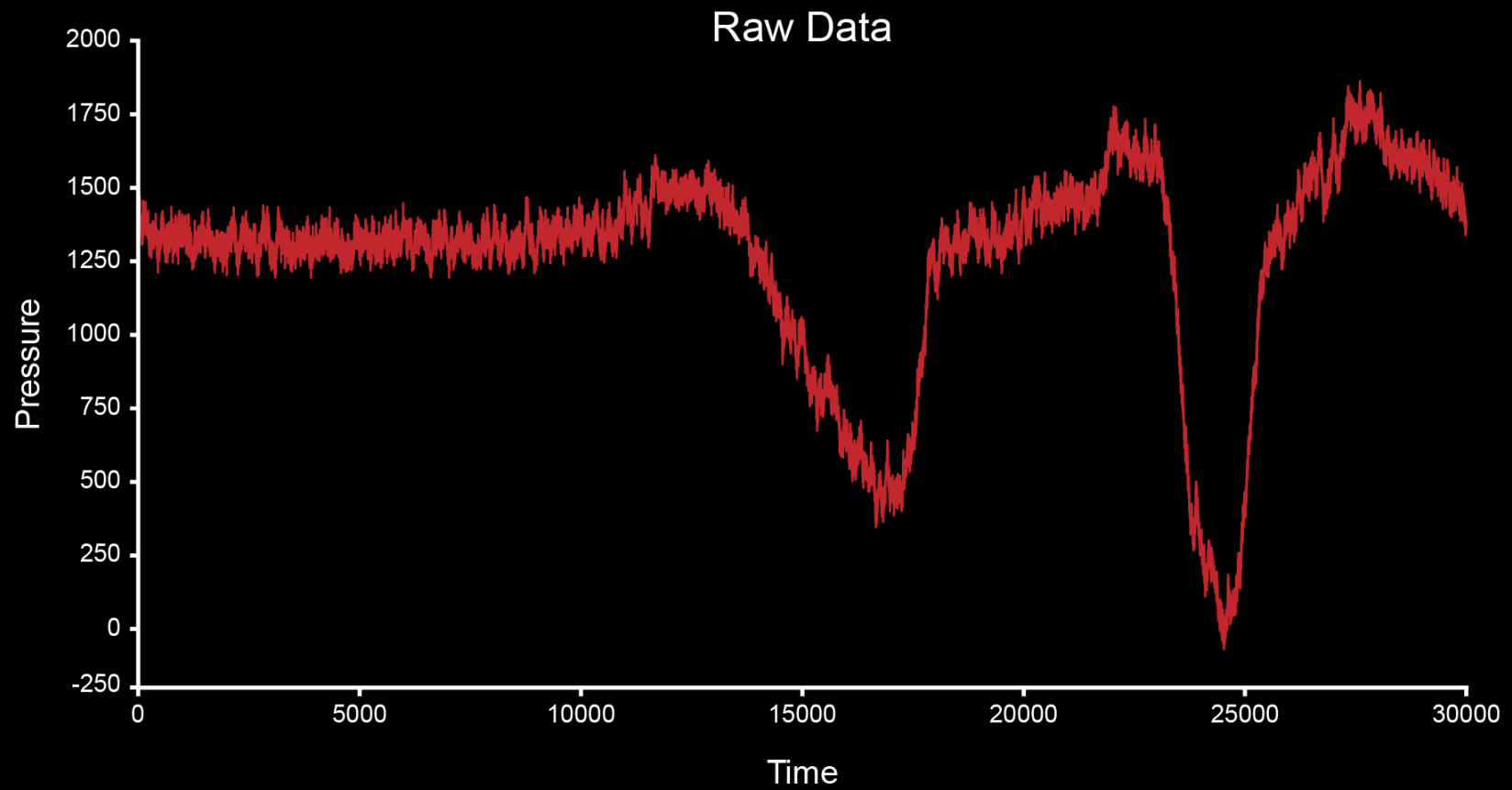


Motivating Example: removing noise

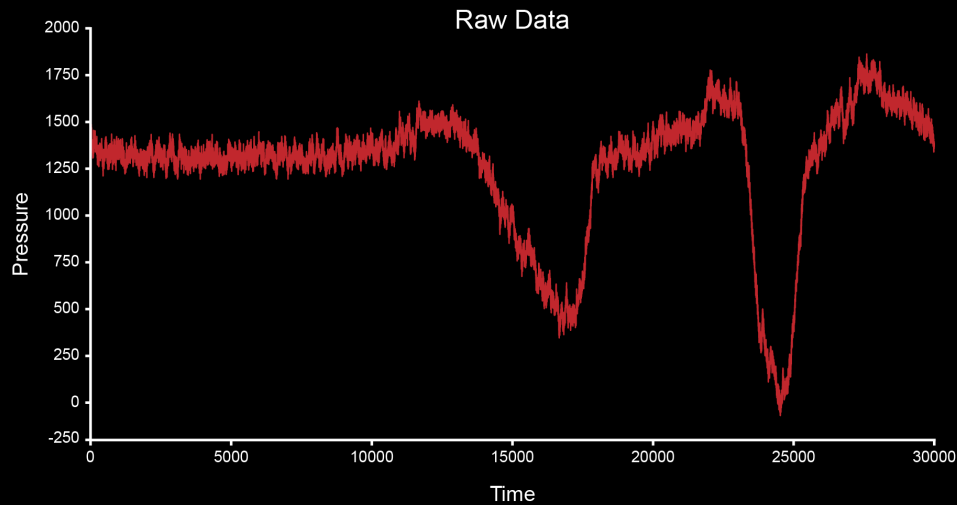


How do we process signals?

How do we process signals?

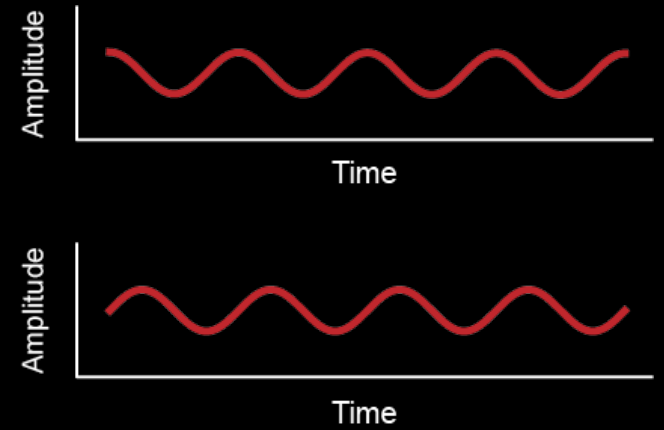
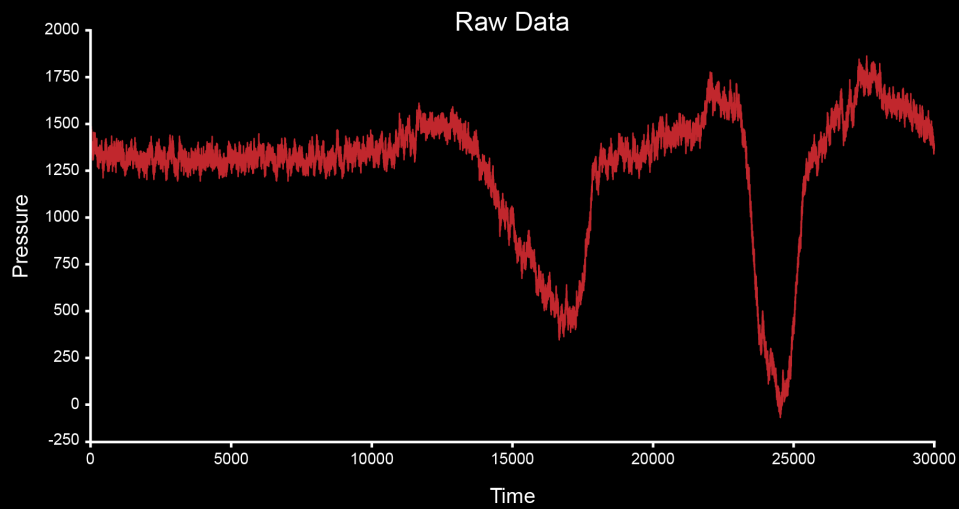


How do we process signals?



Simpler signals
that are easier to
process

How do we process signals?



Signals are combinations of signals...

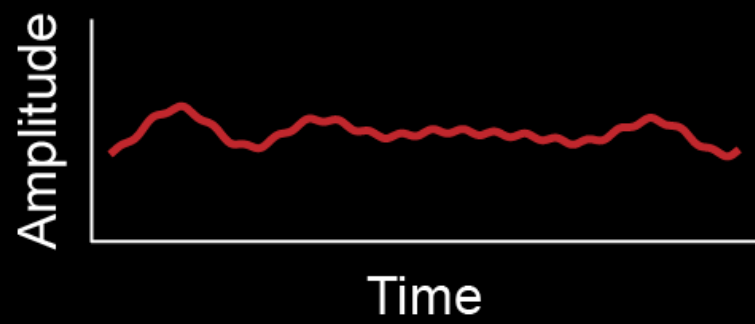
- We can create any signal using a sum of sines and cosines
- Trust me
- Or trust 3blue1brown:

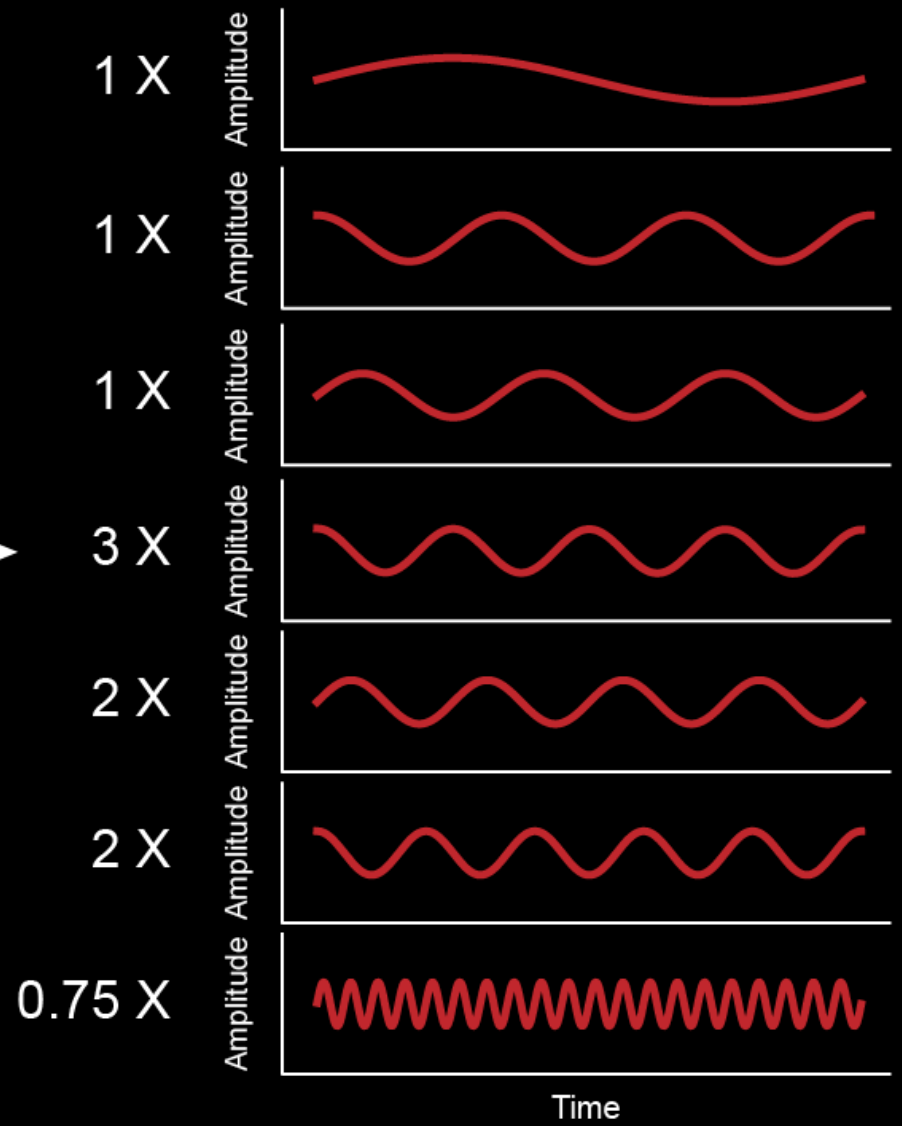
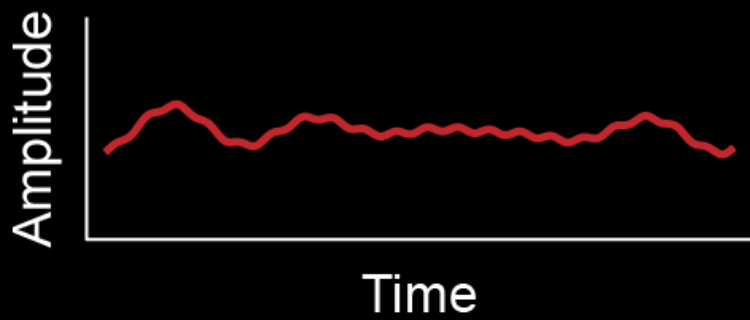
Signals are combinations of signals...

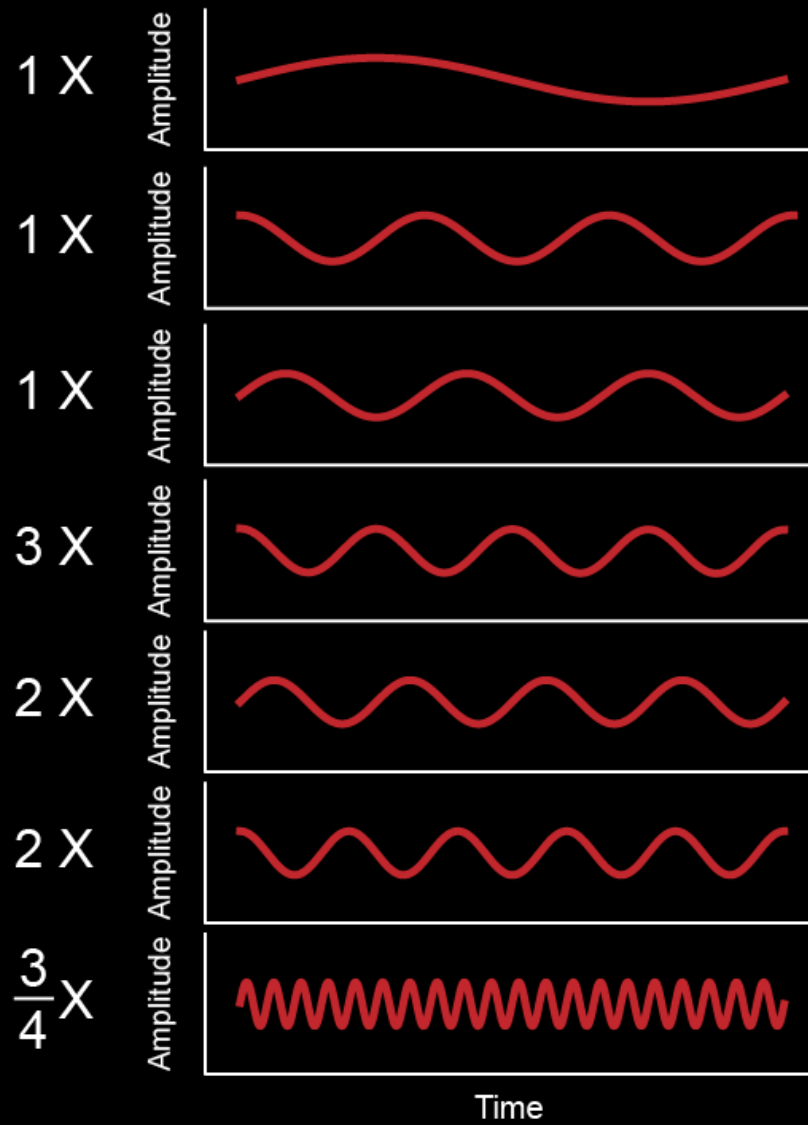
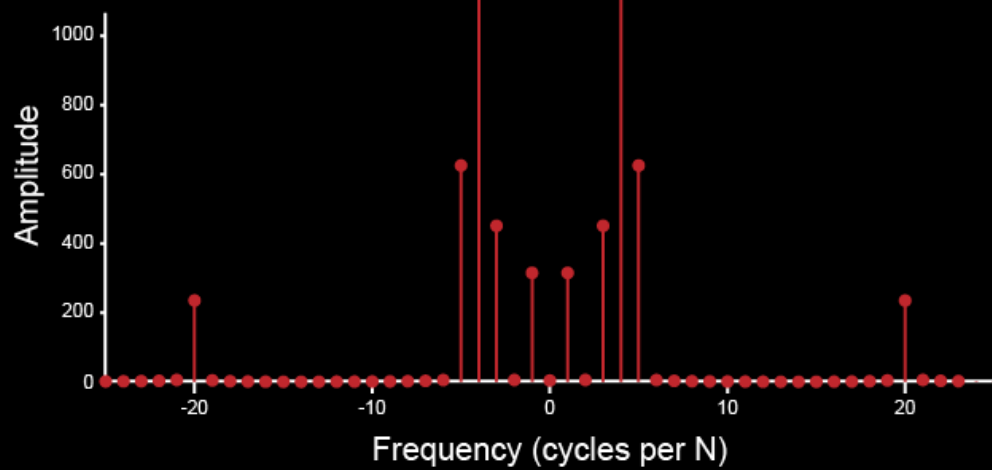


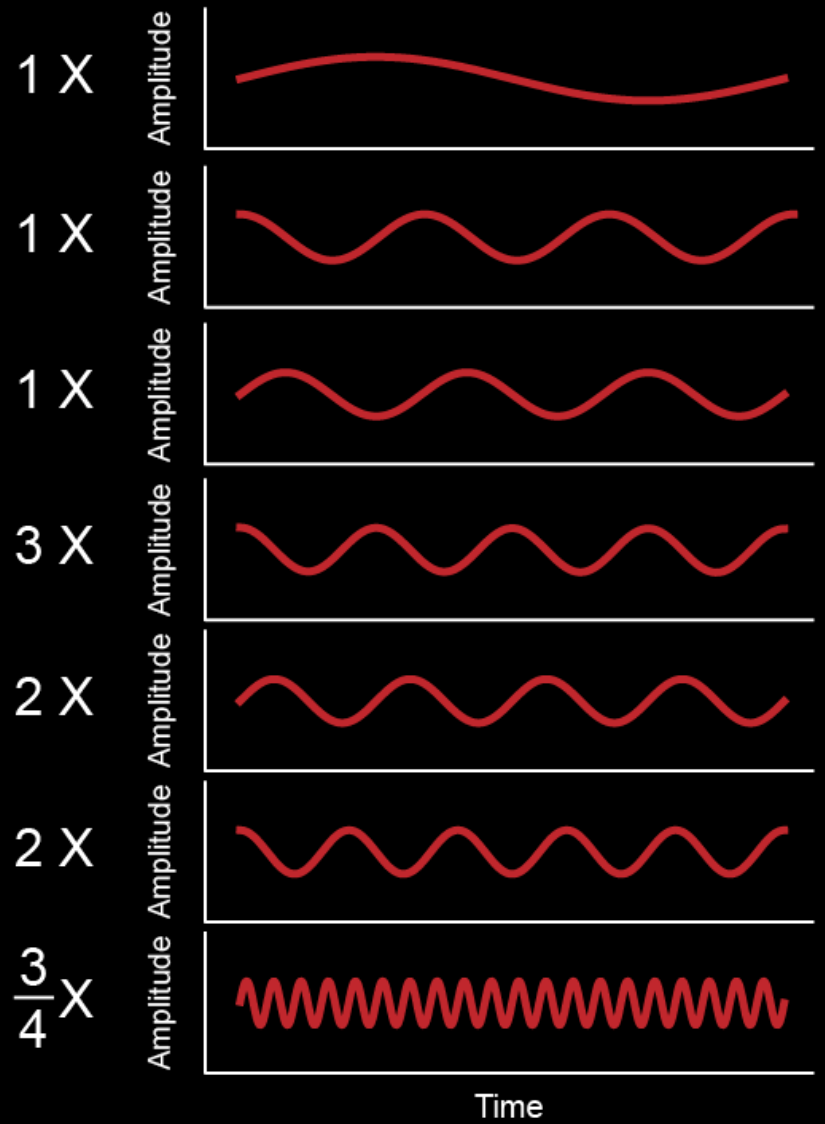
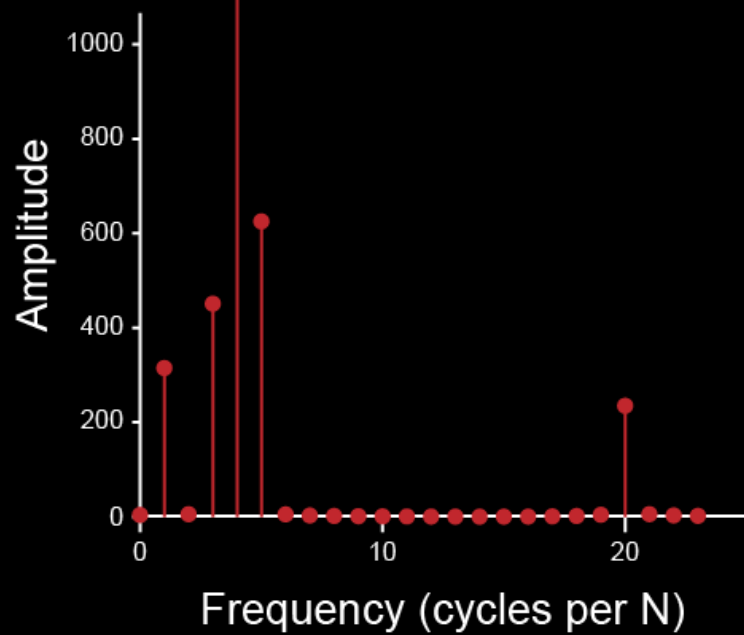
<https://www.youtube.com/watch?v=r6sGWTCMz2k>

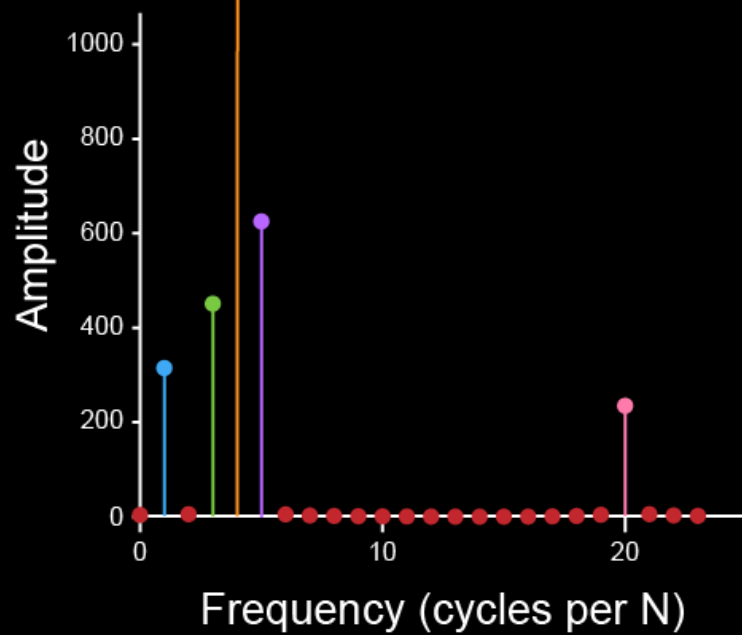
- 5:11-5:50
- 0:34-1:36



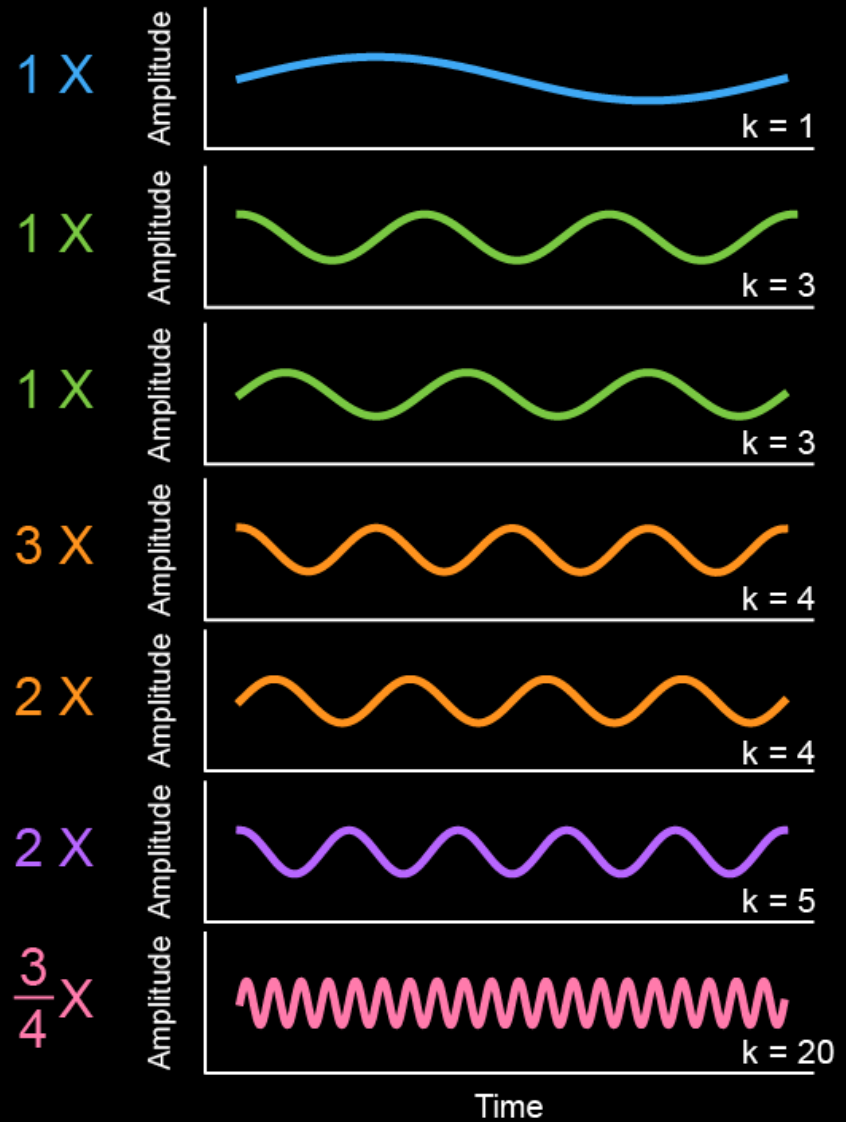


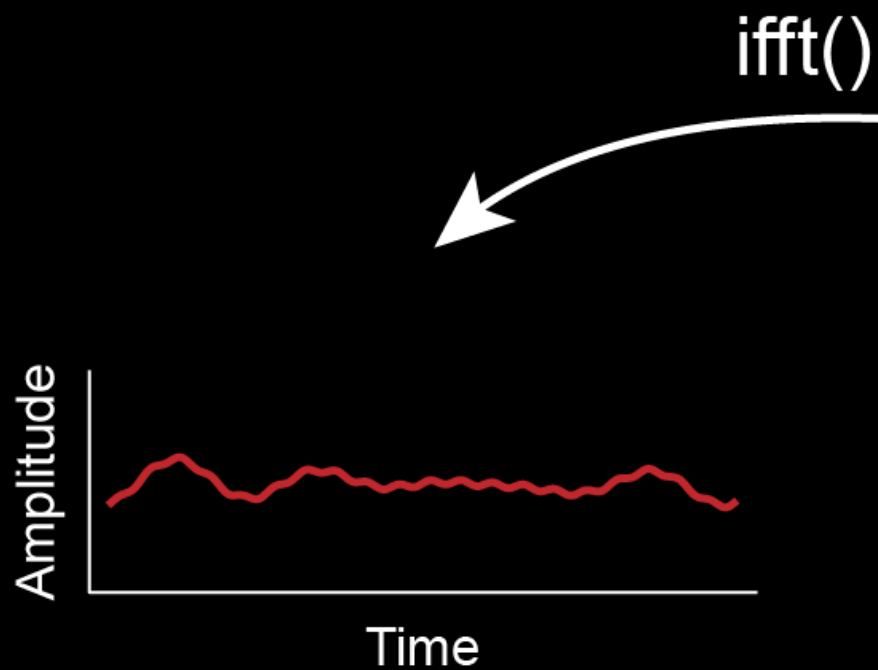






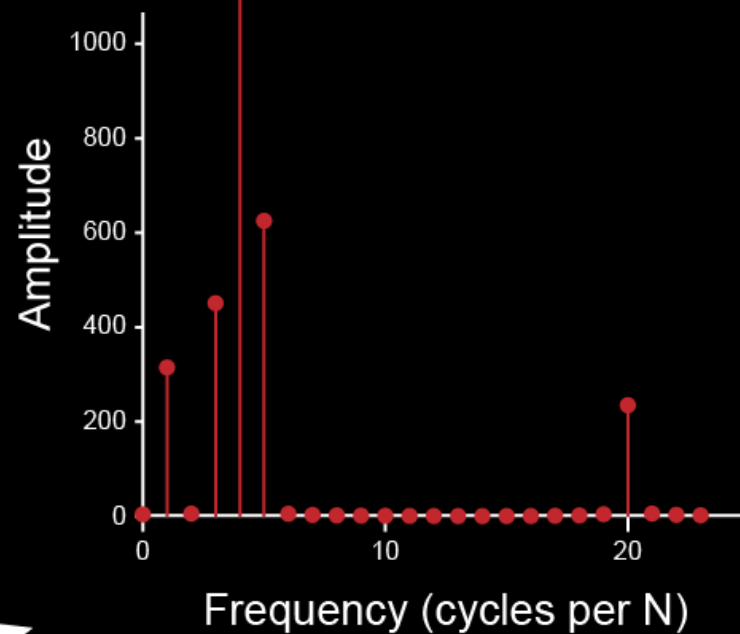
`abs(fft)`





Time Domain

$\text{ifft}()$




Frequency Domain

$\text{fft}()$



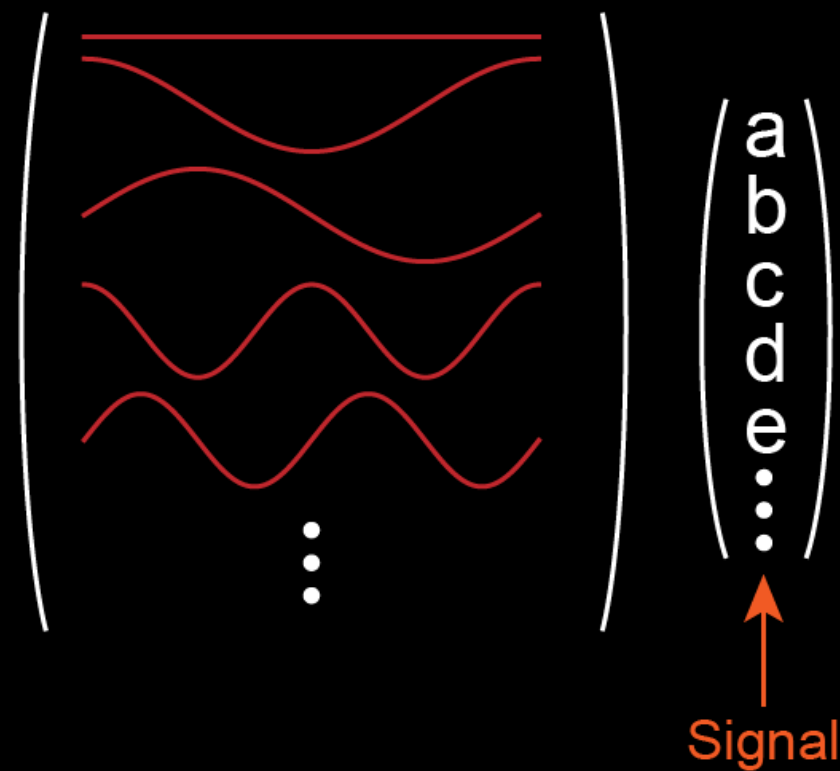
How do you perform the fft or ifft?

How do you perform the fft or ifft?

$$F = \left(\begin{array}{c|c|c|c|c|c} \text{white} & \text{red} & \text{red} & \text{red} & \text{red} & \text{white} \end{array} \right)$$


The diagram illustrates a matrix F used for Fast Fourier Transform (FFT) or Inverse Fast Fourier Transform (IFFT). The matrix is represented by a large white left parenthesis followed by a large white right parenthesis. Inside, there are six columns. The first and last columns are white vertical lines. The four columns in between are red sine waves of increasing frequency from left to right. The first red column has one full cycle, the second has two, the third has three, and the fourth has four. To the right of the fourth red column, there are three white dots, indicating that the matrix continues with more columns of higher frequencies.

How do you perform the fft or ifft?

$$\text{fft}(\vec{r}) = F^T * \vec{r} =$$


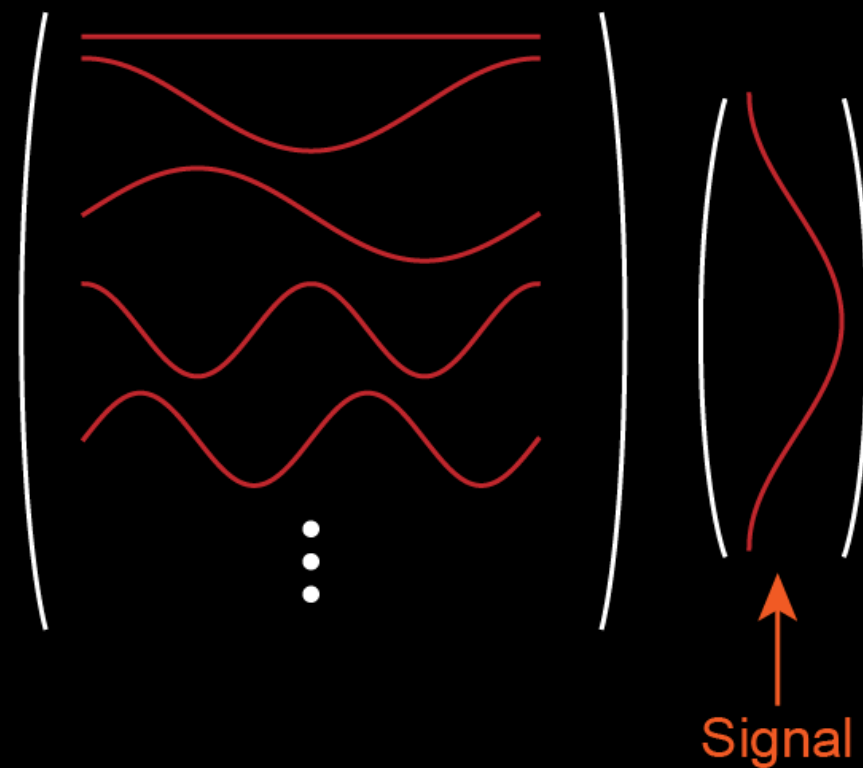
The diagram illustrates the Fast Fourier Transform (FFT) operation. It shows a large matrix of basis functions (rows) multiplied by an input vector \vec{r} . The basis functions are represented by red wavy lines of varying frequencies. The input vector \vec{r} is a column vector with elements a, b, c, d, e , and an ellipsis. An orange arrow labeled "Signal" points to the element e in the input vector.

How do you perform the fft or ifft?

$$\text{fft}(\vec{r}) = F^T * \vec{r} = \begin{pmatrix} \text{---} \bullet (a \ b \ c \ d \ e \dots) \\ \text{---} \bullet (a \ b \ c \ d \ e \dots) \\ \text{---} \bullet (a \ b \ c \ d \ e \dots) \\ \text{---} \bullet (a \ b \ c \ d \ e \dots) \\ \text{---} \bullet (a \ b \ c \ d \ e \dots) \\ \vdots \end{pmatrix}$$

↑
Signal

How do you perform the fft or ifft?

$$\text{fft}(\vec{r}) = F^T * \vec{r} =$$


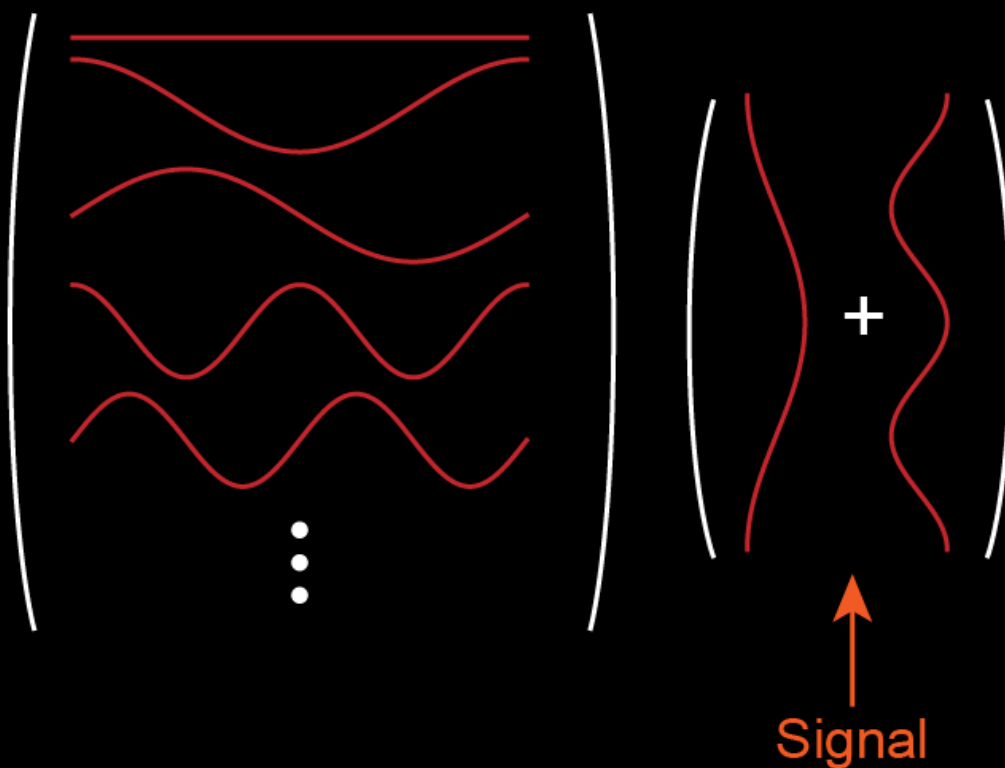
The diagram illustrates the Fast Fourier Transform (FFT) operation. It shows a large matrix of basis functions (rows) multiplied by a signal vector (column). The basis functions are represented by red wavy lines of varying frequencies. The signal vector is represented by a single red wavy line. An orange arrow points to the signal vector with the label "Signal".

How do you perform the fft or ifft?

$$\text{fft}(\vec{r}) = F^T * \vec{r} = \left(\begin{array}{c} \text{---} \cdot \left(\text{---} \right) \\ \text{---} \cdot \left(\text{---} \right) \\ \text{---} \cdot \left(\text{---} \right) \\ \text{---} \cdot \left(\text{---} \right) \\ \text{---} \cdot \left(\text{---} \right) \\ \vdots \end{array} \right) = \left(\begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{array} \right)$$

↑
Signal

How do you perform the fft or ifft?

$$\text{fft}(\vec{r}) = F^T * \vec{r} =$$


The diagram illustrates the Fast Fourier Transform (FFT) operation. It shows a large matrix of basis functions (rows) multiplied by a vector of input samples (columns). The basis functions are represented by red wavy lines of varying frequencies. The input vector is represented by a single red wavy line. The result is a vector of complex coefficients, represented by a single red wavy line. An orange arrow points to the input vector with the label "Signal".

How do you perform the fft or ifft?

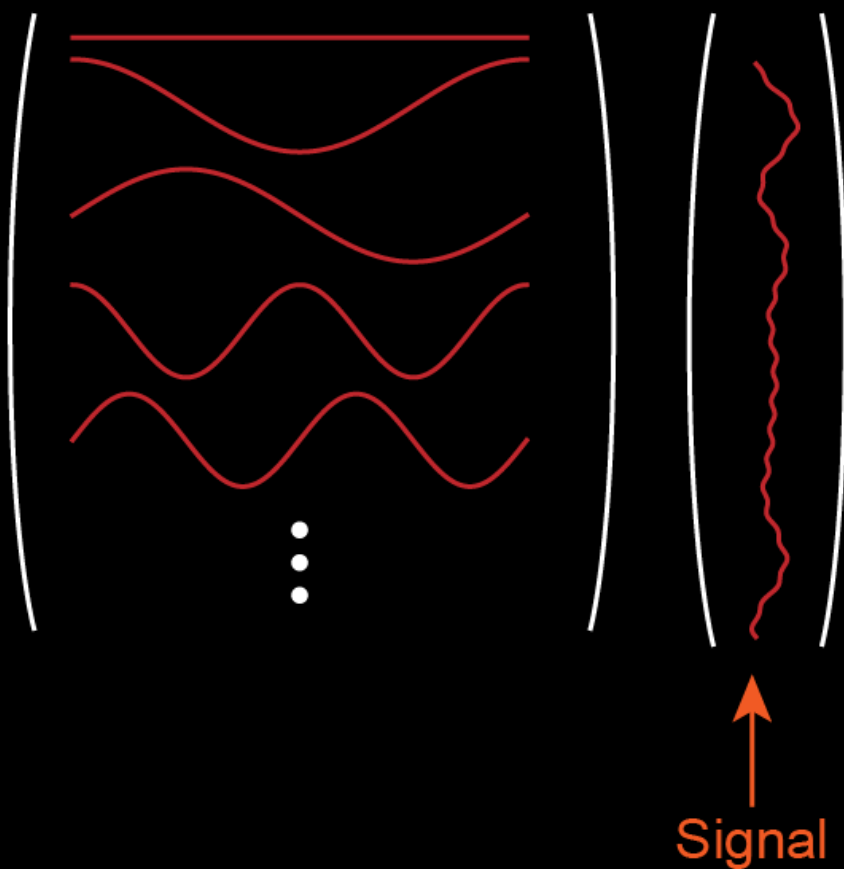
The diagram illustrates the computation of the Discrete Fourier Transform (DFT) using the butterfly network method. It shows a large vector of products of input signals and basis functions, summed together, resulting in a vector of zeros and ones.

The input signal is represented by a horizontal line at the bottom, labeled "Signal". Two orange arrows point upwards from this line to the two main groups of products in the large vector.

The large vector is composed of two main groups of products, each enclosed in large parentheses. Each group contains five rows of products, with a vertical ellipsis indicating more rows. Each row consists of an input signal (red or blue wavy line) multiplied by a basis function (red wavy line), followed by a plus sign and another product of the same input signal and basis function.

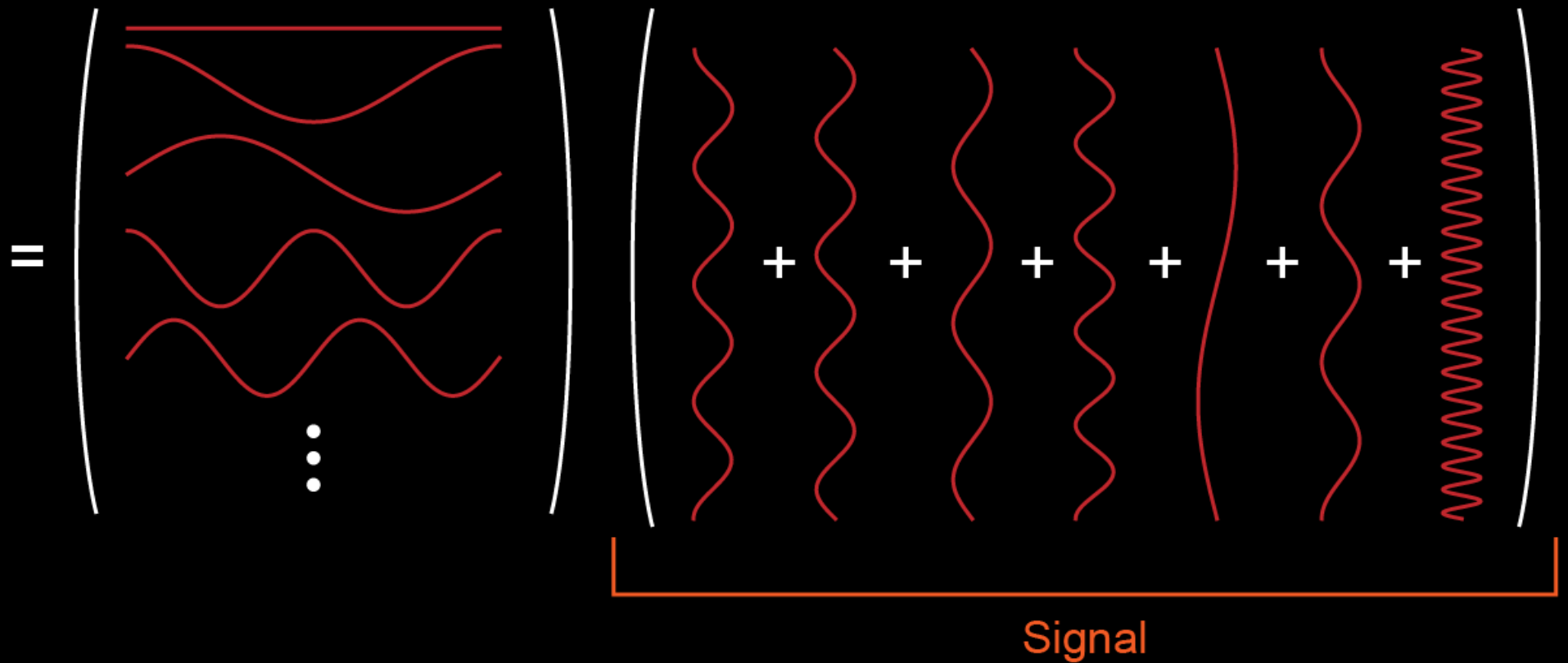
The resulting vector, indicated by an equals sign, is a column vector containing the values 0, 1, 0, 1, 0, and a vertical ellipsis, representing the DFT coefficients.

How do you perform the fft or ifft?

$$\text{fft}(\vec{r}) = F^T * \vec{r} =$$


The diagram illustrates the Fast Fourier Transform (FFT) operation. On the left, a large white curved bracket encloses a vertical stack of red lines representing basis functions: a flat line, a single cycle of a cosine wave, a single cycle of a sine wave, and a higher-frequency wave, followed by three vertical dots. To the right of this is a smaller white curved bracket containing a single vertical red wavy line. An orange arrow points from the word "Signal" below to this wavy line.

How do you perform the fft or ifft?




How do you perform the fft or ifft?

$$\text{ifft}(\tilde{r}) = F^* \tilde{r} = \left(\begin{array}{c} | \\ \text{wavy} \\ \text{wavy} \\ \text{wavy} \\ \dots \end{array} \right) \begin{pmatrix} a \\ b \\ c \\ d \\ \vdots \end{pmatrix}$$

$$= a \begin{array}{c} | \\ \text{wavy} \end{array} + b \begin{array}{c} \text{wavy} \end{array} + c \begin{array}{c} \text{wavy} \end{array} + d \begin{array}{c} \text{wavy} \end{array} + e \begin{array}{c} \text{wavy} \end{array} \dots$$

How do you perform the fft or ifft?

$$F = \left(\begin{array}{c|c|c|c|c|c} \text{white} & \text{red} & \text{red} & \text{red} & \text{red} & \text{white} \end{array} \right)$$


The diagram shows a matrix F enclosed in large white parentheses. The matrix is composed of several columns. The first column is a solid vertical red line. The subsequent columns are red curves that represent increasing frequencies: a single cycle, two cycles, three cycles, and four cycles. The matrix is followed by an ellipsis (...) and then a final white column, indicating a sequence of basis functions.

How do you perform the fft or ifft?

$$F = \begin{pmatrix} k=0 & k=1 & k=2 & k=3 & \dots \\ \text{vertical line} & \text{sin wave} & \text{cos wave} & \text{sin wave} & \dots \end{pmatrix}$$

$$e^{\frac{i * 2\pi k * n}{N}} = \cos\left(\frac{2\pi k}{N} n\right) + i * \sin\left(\frac{2\pi k}{N} n\right)$$

How do you *actually* perform the fft?

Signal (N x 1)



$$1) \quad \tilde{r} = \text{fft}(\vec{r})$$



Frequency components (N x 1, complex)

1) $\tilde{r} = \text{fft}(\vec{r})$

(i) $\tilde{r} = \text{real}(\text{fft}(\vec{r}))$

(ii) $\tilde{r} = \text{imag}(\text{fft}(\vec{r}))$

(iii) $\tilde{r} = \text{abs}(\text{fft}(\vec{r}))$

(iv) $\tilde{r} = \text{angle}(\text{fft}(\vec{r}))$

$$e^{\frac{i * 2\pi k * n}{N}} = \cos\left(\frac{2\pi k}{N} n\right) + i * \sin\left(\frac{2\pi k}{N} n\right)$$

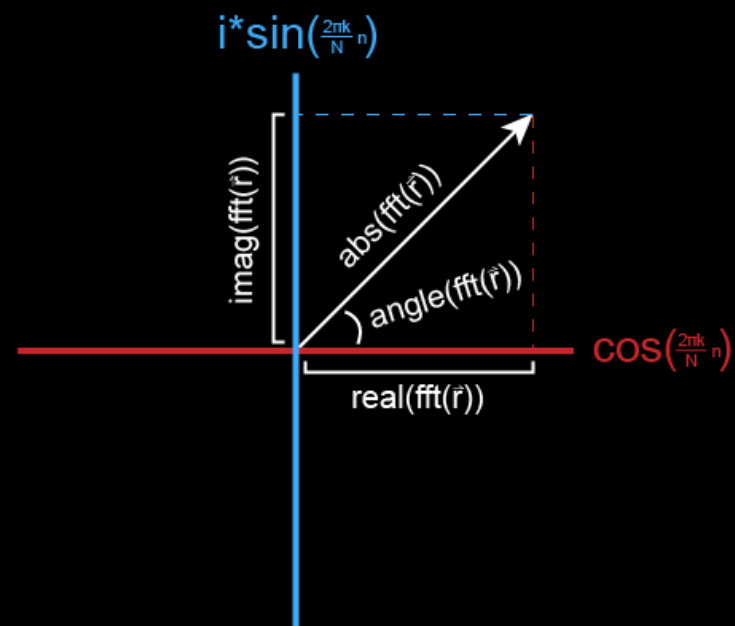
$$1) \tilde{r} = \text{fft}(\vec{r})$$

$$(i) \tilde{r} = \text{real}(\text{fft}(\vec{r}))$$

$$(ii) \tilde{r} = \text{imag}(\text{fft}(\vec{r}))$$

$$(iii) \tilde{r} = \text{abs}(\text{fft}(\vec{r}))$$

$$(iv) \tilde{r} = \text{angle}(\text{fft}(\vec{r}))$$



1) $\tilde{r} = \text{fft}(\vec{r})$

Frequency components (N x 1, complex)

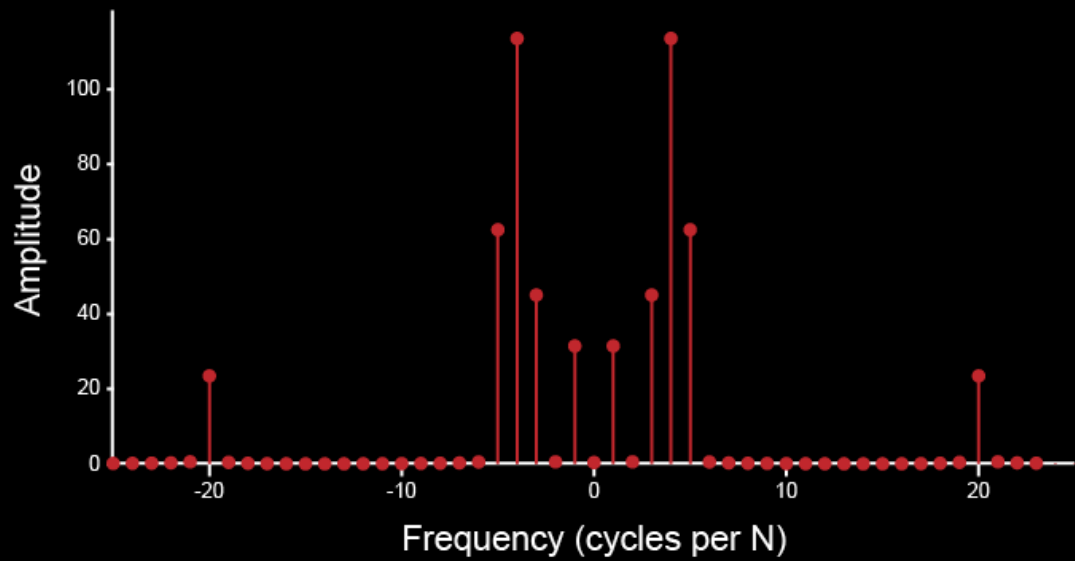
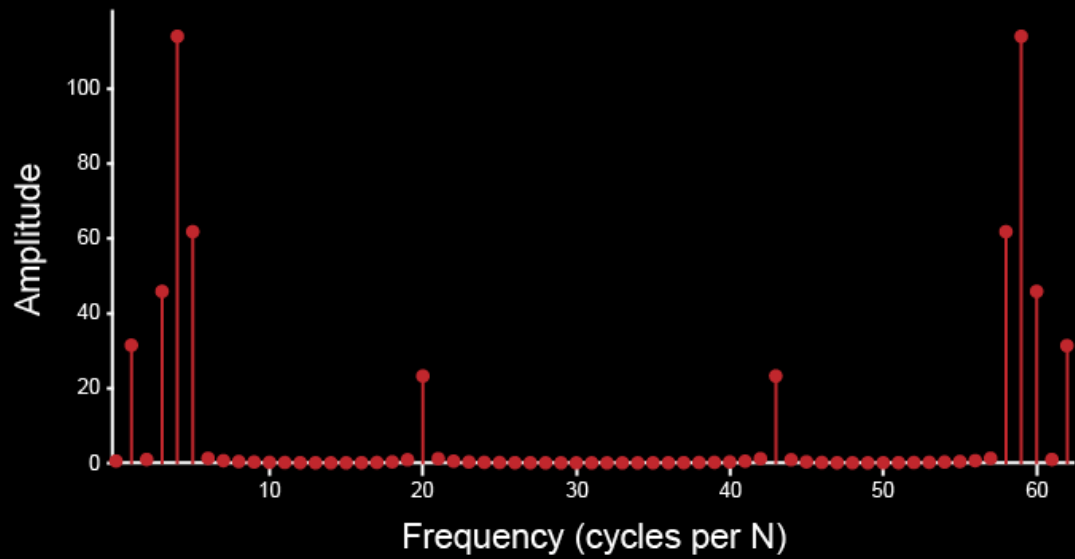


2) $\tilde{r}_{\text{shifted}} = \text{fftshift}(\tilde{r})$



Frequency components (N x 1, complex)
Rearranged to make interpretation easier

`fftshift(r)`




1) $\tilde{r} = \text{fft}(\vec{r})$


2) $\tilde{r}_{\text{shifted}} = \text{fftshift}(\tilde{r})$

3) $\text{stem}(x, \tilde{r}_{\text{shifted}})$

Frequencies
-N/2 -> N/2



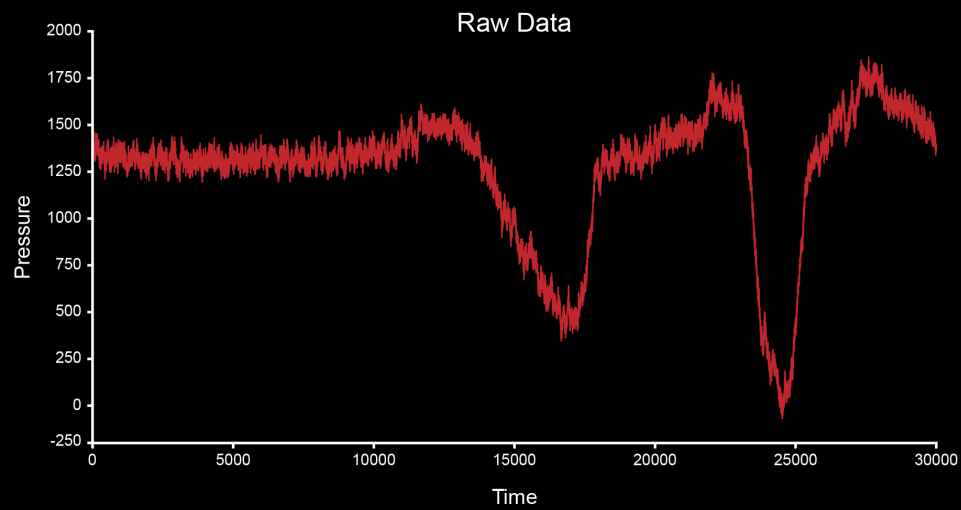
Frequency components (N x 1, complex)
Rearranged to make interpretation easier



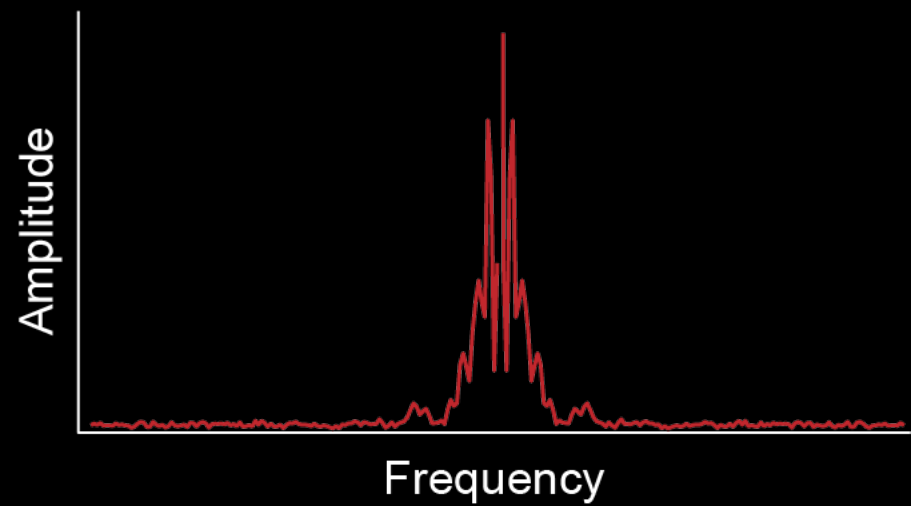
Exercise #1

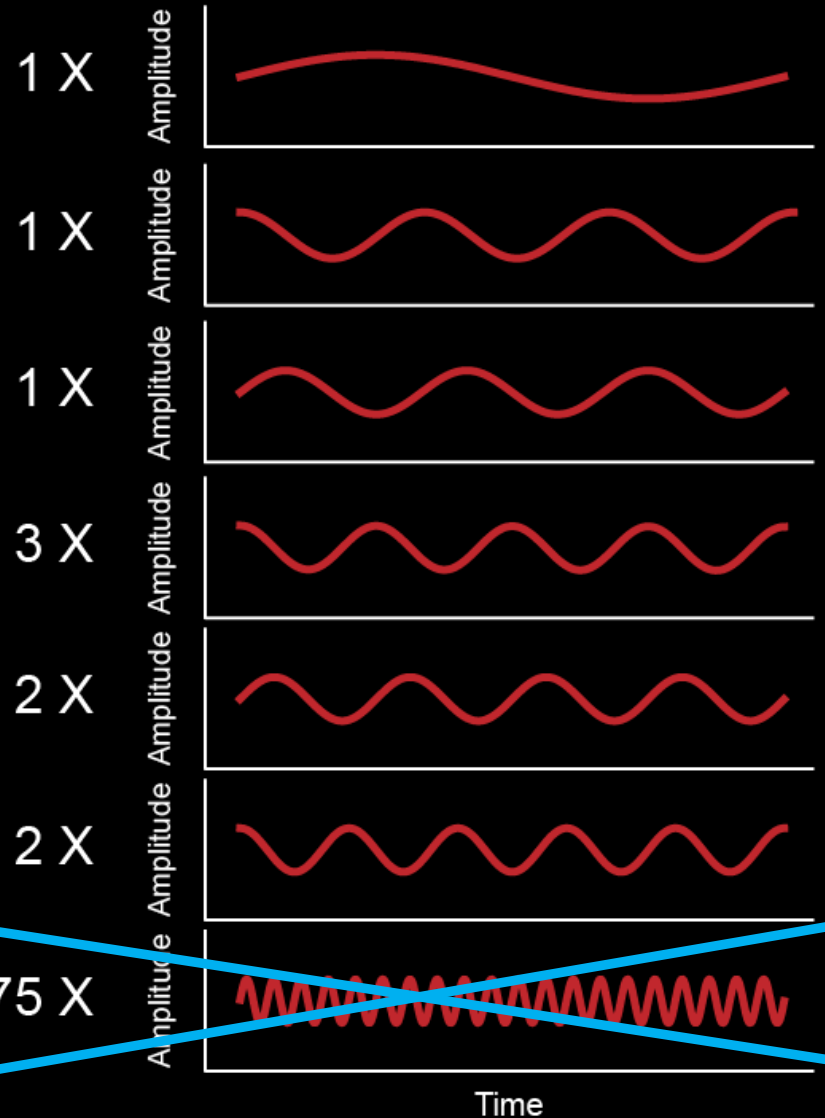
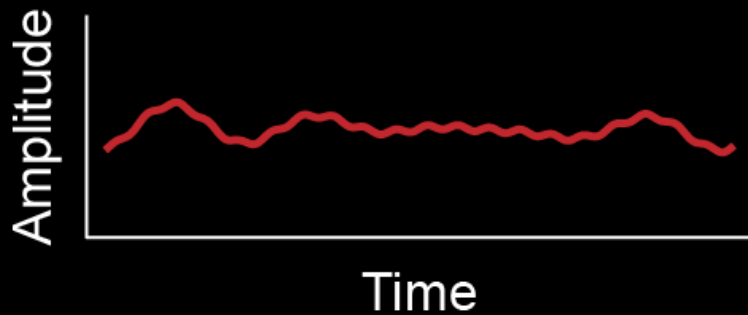
Fourier Transform and Shift-Invariant Systems

What now?



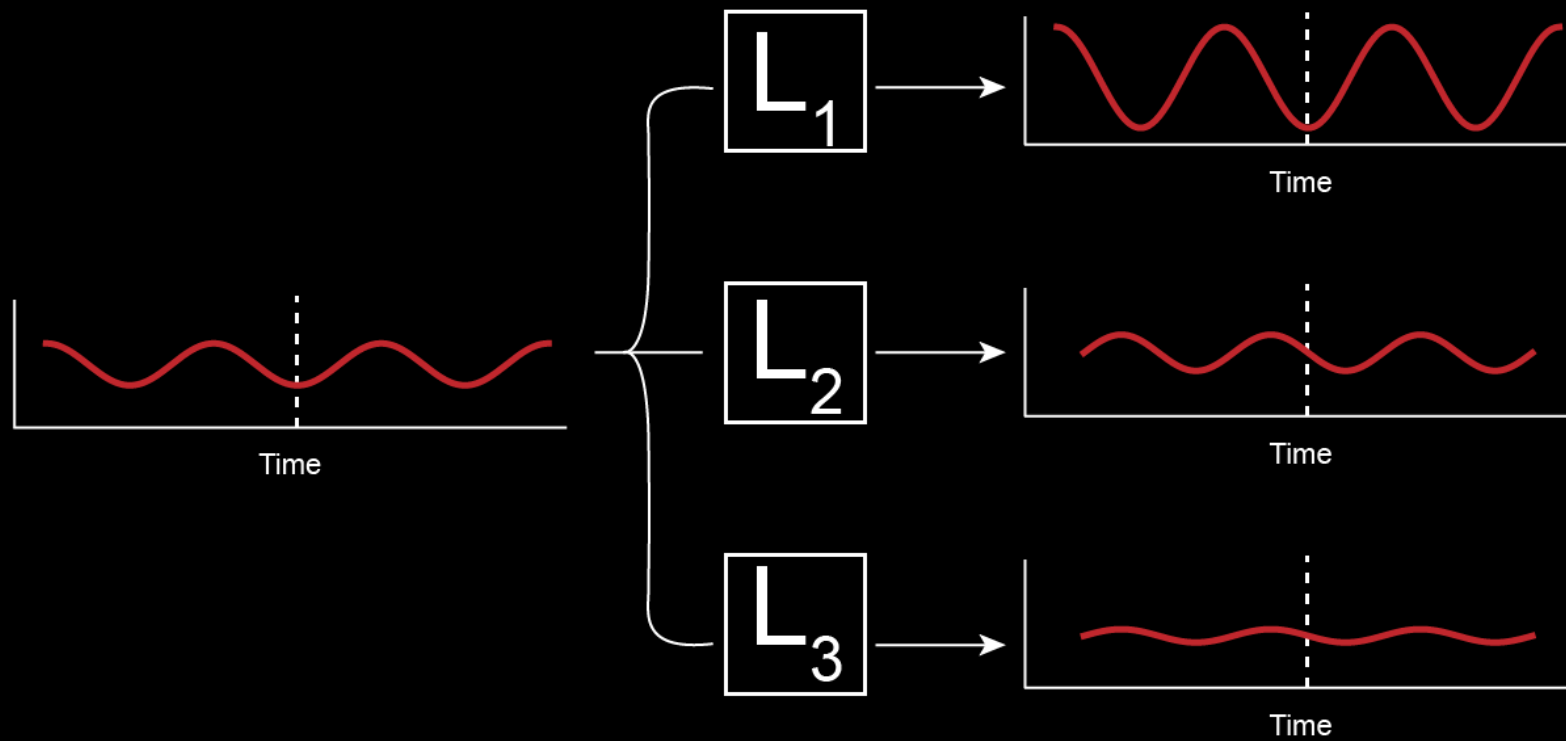
FFT





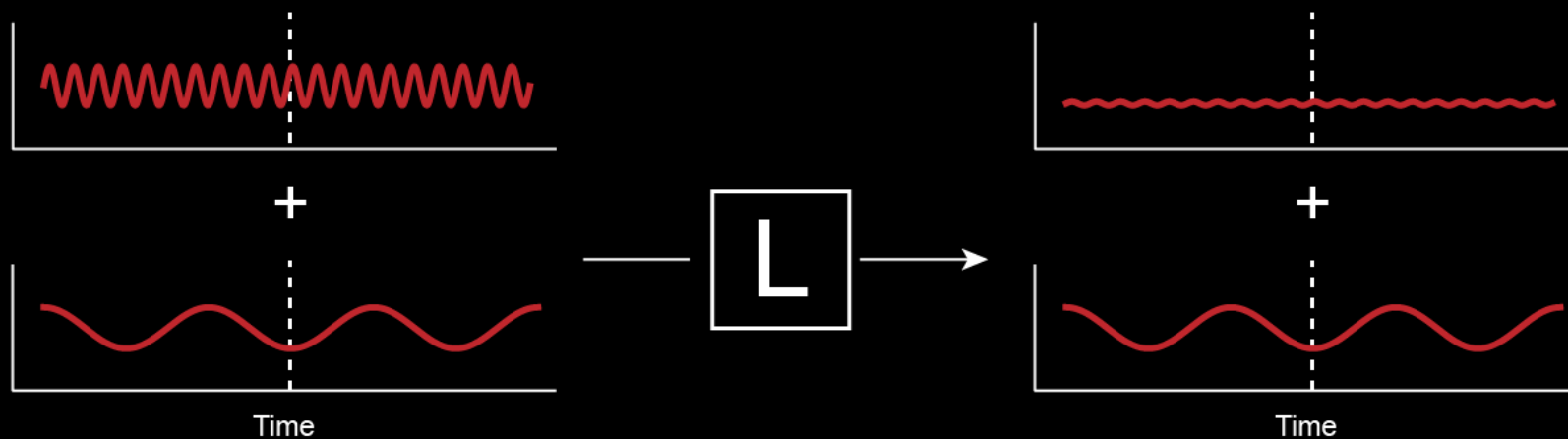
The entire point of this is to process our signals. Is there a way to remove or modify certain waves but not others?

Is there a way to remove or modify certain waves but not others?



Linear shift invariant systems

Is there a way to remove or modify certain waves but not others?



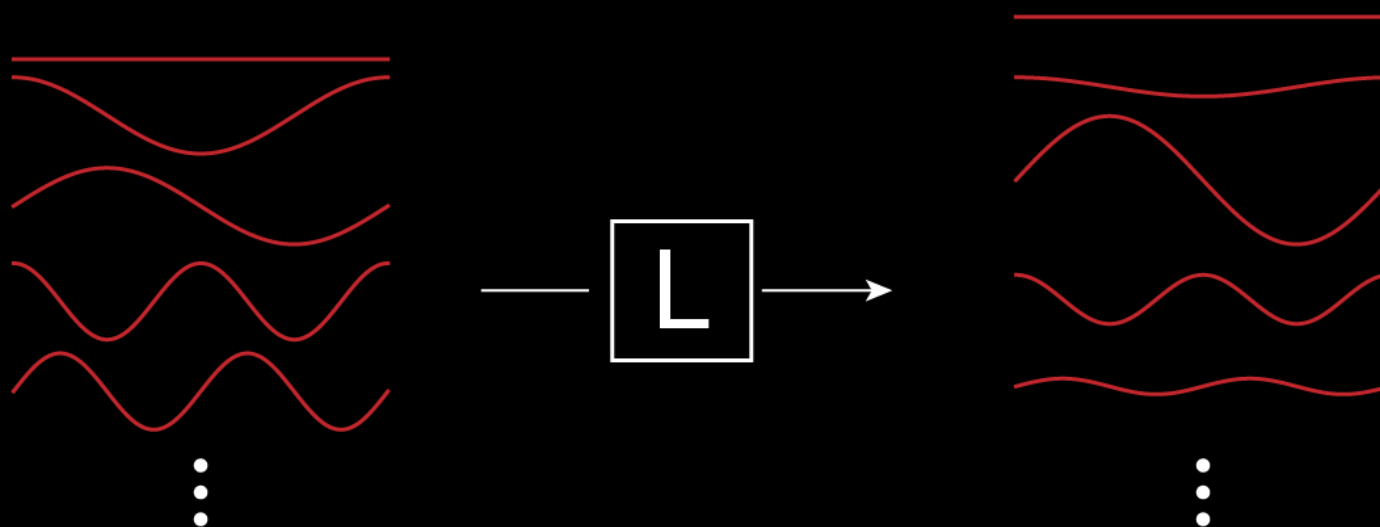
Linear shift invariant system

How do we use linear shift invariant systems to
analyze our data?

How do we use linear shift invariant systems to analyze our data?

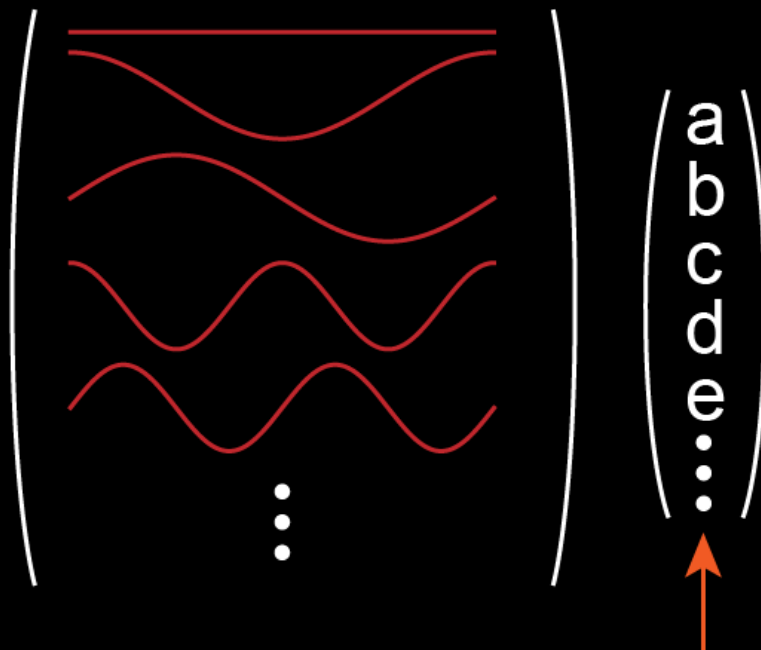
1. Figure out how the system will alter cosine and sine waves

How do we use linear shift invariant systems to analyze our data?



Linear shift invariant system

How do we use linear shift invariant systems to analyze our data?

$$\text{fft}(\vec{r}) = F^T * \vec{r} =$$


Impulse response / kernel

How do we use linear shift invariant systems to analyze our data?

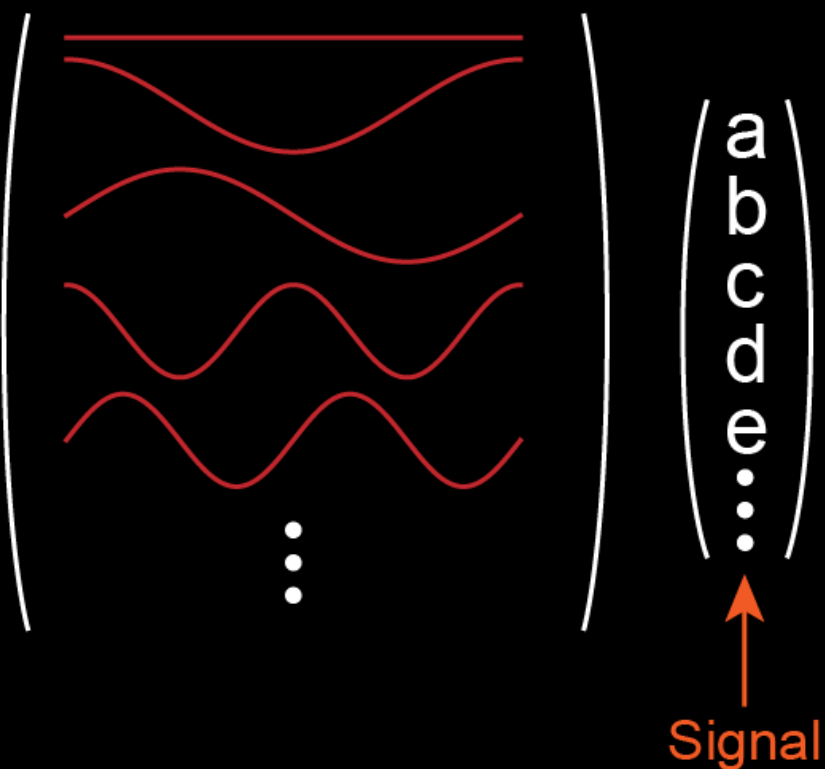
$$\text{fft}(\vec{r}) = F^T * \vec{r} = \begin{pmatrix} \text{---} \cdot (a \ b \ c \ d \ e \dots) \\ \text{---} \cdot (a \ b \ c \ d \ e \dots) \\ \text{---} \cdot (a \ b \ c \ d \ e \dots) \\ \text{---} \cdot (a \ b \ c \ d \ e \dots) \\ \text{---} \cdot (a \ b \ c \ d \ e \dots) \\ \vdots \end{pmatrix} = \begin{pmatrix} C_r(\omega_{k=0}) \\ C_r(\omega_{k=1}) \\ S_r(\omega_{k=1}) \\ C_r(\omega_{k=2}) \\ S_r(\omega_{k=2}) \\ \vdots \end{pmatrix}$$

↑
Impulse response / kernel

How do we use linear shift invariant systems to analyze our data?

1. Figure out how the system will alter cosine and sine waves
 - Take `fft()` of impulse response
2. Figure out how your signal of interest is broken down into cosine and sine waves

How do we use linear shift invariant systems to analyze our data?

$$\text{fft}(\hat{\mathbf{x}}) = \mathbf{F}^T * \hat{\mathbf{x}} =$$


The diagram illustrates the Fast Fourier Transform (FFT) process. It shows a large matrix of red sine waves, representing the Fourier transform basis functions, enclosed in large white parentheses. To the right of this matrix is a vertical vector of letters: 'a', 'b', 'c', 'd', 'e', and an ellipsis, also enclosed in large white parentheses. An orange arrow labeled 'Signal' points to the letter 'e' in the vector, indicating the input signal being analyzed.

How do we use linear shift invariant systems to analyze our data?

1. Figure out how the system will alter cosine and sine waves
 - Take `fft()` of impulse response
2. Figure out how your signal of interest is broken down into cosine and sine waves
 - Take `fft()` of signal
3. Use #1 and #2 to figure out how the system alters your signal

How do we use linear shift invariant systems to analyze our data?

$$\begin{array}{c} \tilde{\mathbf{R}} \end{array} \begin{pmatrix} C_r(\omega_{k=0}) & 0 & 0 & 0 & 0 \\ 0 & C_r(\omega_{k=1}) & 0 & 0 & 0 \\ 0 & 0 & S_r(\omega_{k=1}) & 0 & 0 \\ 0 & 0 & 0 & C_r(\omega_{k=2}) & 0 \\ 0 & 0 & 0 & 0 & S_r(\omega_{k=2}) \dots \\ \vdots & & & & \end{pmatrix} \begin{array}{c} \text{fft}(\vec{x}) \end{array} \begin{pmatrix} C_x(\omega_{k=1}) \\ S_x(\omega_{k=1}) \\ C_x(\omega_{k=2}) \\ S_x(\omega_{k=2}) \\ \vdots \end{pmatrix}$$

What happens to each cosine and sine wave
in the form of a diagonal matrix

Amount of each cosine and sine
wave in signal

How do we use linear shift invariant systems to analyze our data?

1. Figure out how the system will alter cosine and sine waves
 - Take `fft()` of impulse response
2. Figure out how your signal of interest is broken down into cosine and sine waves
 - Take `fft()` of signal
3. Use #1 and #2 to figure out how the system alters your signal
 - $R * \text{fft}(x)$
4. Convert back into a signal

How do you perform the fft or ifft?

$$\text{ifft}(\tilde{r}) = F^* \tilde{r} = \left(\begin{array}{c} | \\ \text{wavy} \\ \text{wavy} \\ \text{wavy} \\ \dots \end{array} \right) \begin{pmatrix} a \\ b \\ c \\ d \\ \vdots \end{pmatrix}$$

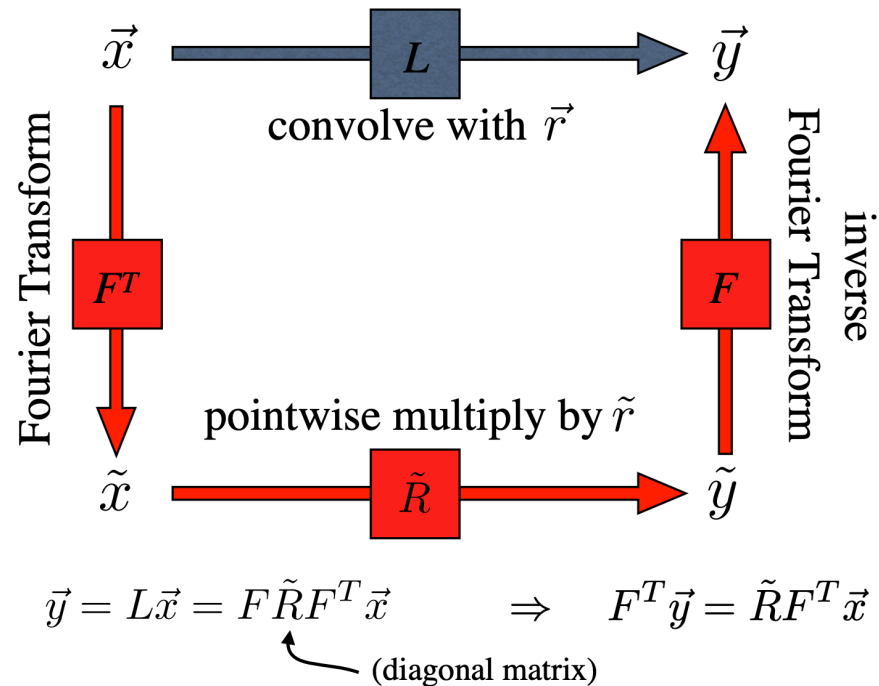
$$= a \begin{array}{c} | \\ \text{wavy} \end{array} + b \begin{array}{c} \text{wavy} \end{array} + c \begin{array}{c} \text{wavy} \end{array} + d \begin{array}{c} \text{wavy} \end{array} + e \begin{array}{c} \text{wavy} \end{array} \dots$$

How do we use linear shift invariant systems to analyze our data?

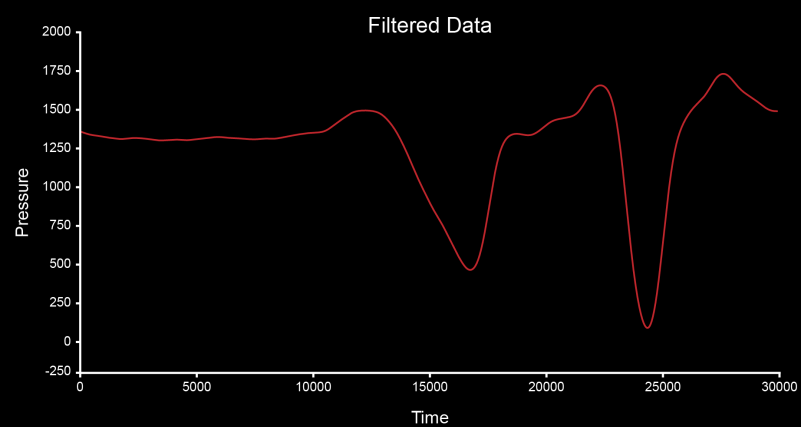
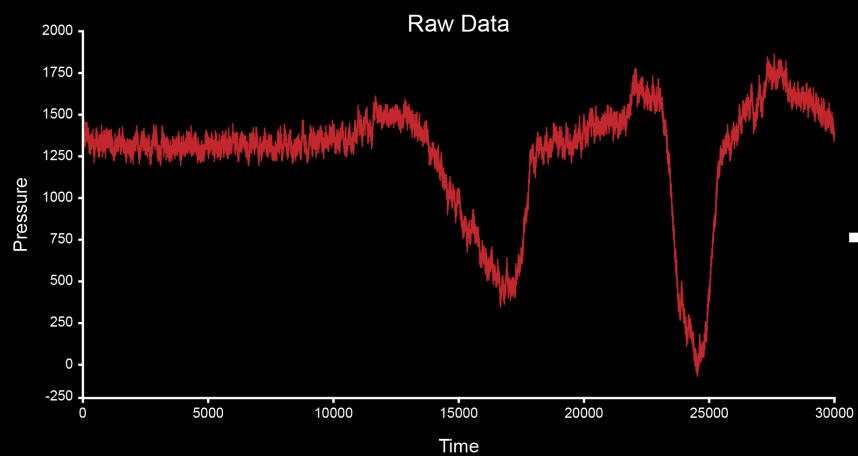
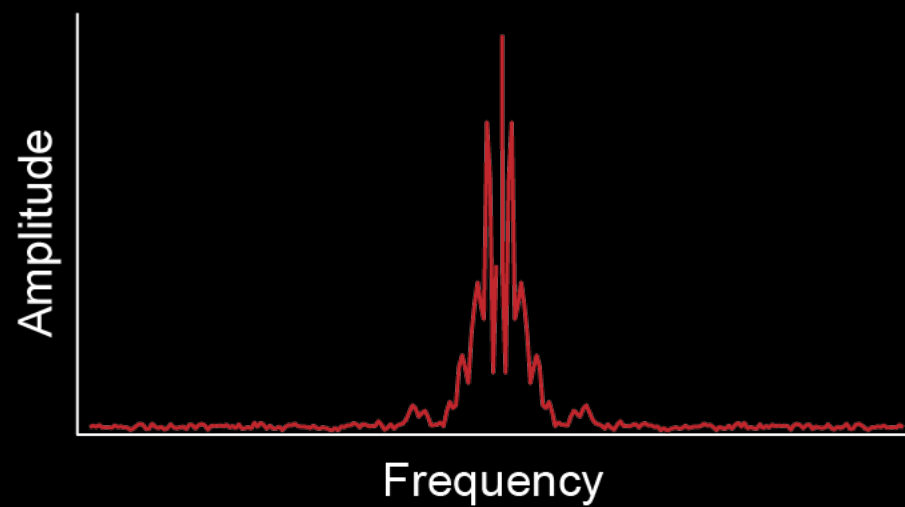
1. Figure out how the system will alter cosine and sine waves
 - Take `fft()` of impulse response
2. Figure out how your signal of interest is broken down into cosine and sine waves
 - Take `fft()` of signal
3. Use #1 and #2 to figure out how the system alters your signal
 - $R * \text{fft}(x)$
4. Convert back into a signal
 - `ifft(R * fft(x))`

How do we use linear shift invariant systems to analyze our data?

The “convolution theorem”



Exercise #2



Questions?