Mathematical Tools
for Neural and Cognitive Science

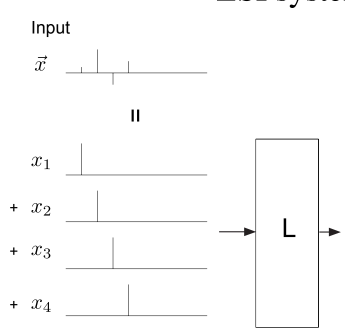
Fall semester, 2020

Section 3:
Linear Shift-Invariant Systems

Linear shift-invariant (LSI) systems

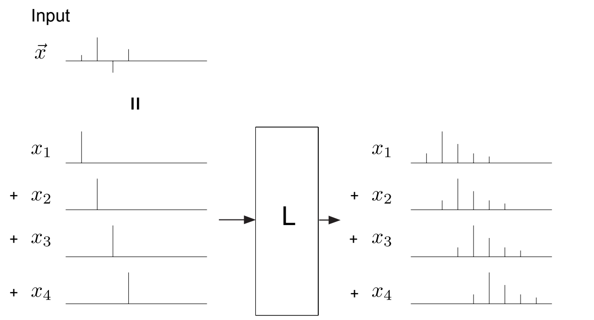
- Linearity (previously discussed):
  "linear combination in, linear combination out"
- Shift-invariance (new property):
  "shifted vector in, shifted vector out"
- These two properties are independent (think of some examples that have both, one, or neither)

LSI system



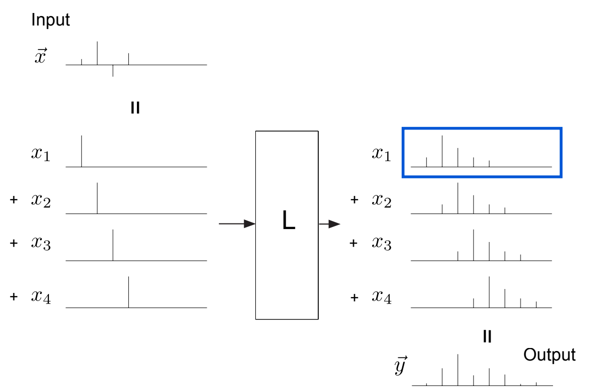
As before, express input as a sum of "impulses", weighted by elements of $\vec{x}$

LSI system



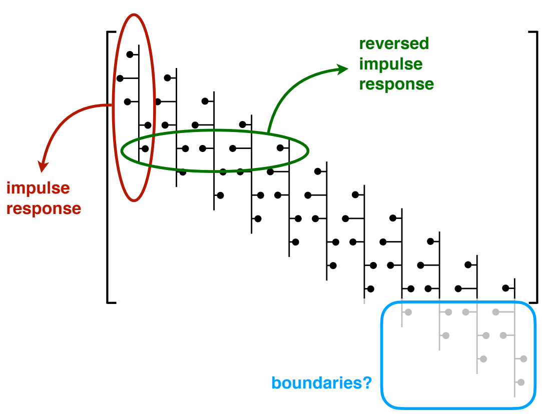
- Linearity => response to $\vec{x}$ is sum of responses to impulses, weighted by elements of $\vec{x}$
- Shift-invariance => responses to impulses are shifted copies of each other

LSI system

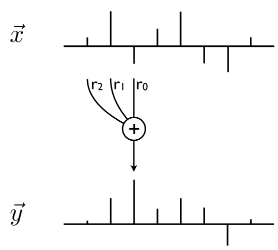


LSI systems are characterized by their "impulse response"

Convolution matrix



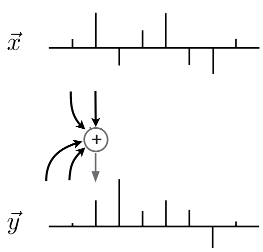
Convolution



$$
\begin{aligned}
y(n) &= \sum_k r(n-k)x(k) \\
&= \sum_k r(k)x(n-k)
\end{aligned}
$$

- Sliding dot product
- Structured matrix
- Boundaries? zero-padding, reflection, circular
- Examples: impulse, delay, average, difference

Feedback LSI system



- Response depends on input, *and* previous outputs
- *Infinite* impulse response (IIR)
- Recursive => possibly *unstable*

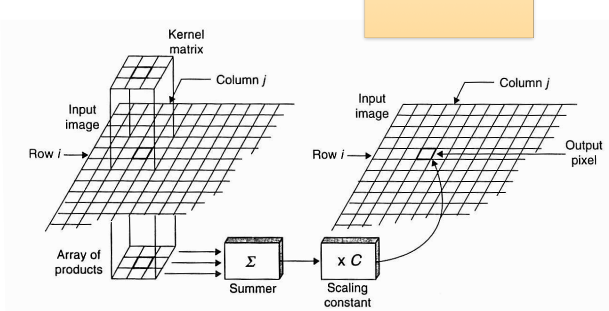
$$
y(n) = \sum_k f(n-k)x(k) + \sum_k g(n-k)y(k)
$$

(For this class, we'll stick to feedforward (FIR) systems)

2D convolution

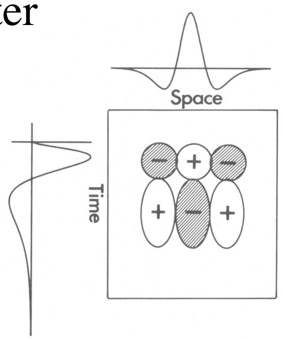
"sliding window"

this figure sucks. Replace it!!



[figure c/o Castleman]

“separable” filter



- Outer product
- Simple design/implementation
- Efficient computation

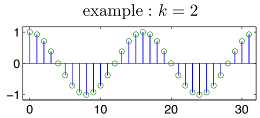
[figure: Adelson & Bergen 85]

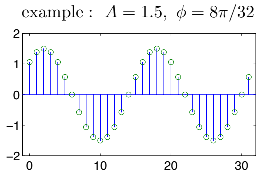
Discrete Sinusoids

“frequency” (cycles/vectorLength)

$\cos(\omega n), \qquad \omega = 2\pi k/N$

“frequency” (radians/sample)





More generally: $A\cos(\omega n - \phi)$

“amplitude”

“phase” (radians)

Shifting Sinusoids

$$A\cos(\omega n - \phi) = A\cos(\phi)\cos(\omega n) + A\sin(\phi)\sin(\omega n)$$

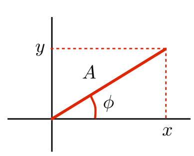
... via a well-known trigonometric identity:

$$\cos(a - b) = \cos(a)\cos(b) + \sin(a)\sin(b)$$

We’ll also need conversions between polar and rectangular coordinates:

$$x = A\cos(\phi), \quad y = A\sin(\phi)$$

$$A = \sqrt{x^2 + y^2}, \quad \phi = tan^{-1}(y/x)$$

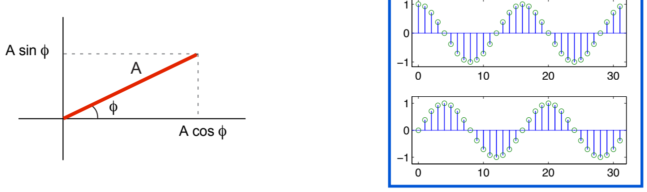


Shifting Sinusoids

$$A\cos(\omega n - \phi) = A\cos(\phi)\cos(\omega n) + A\sin(\phi)\sin(\omega n)$$

scale factors:

fixed cos/sin vectors:



Any *scaled* and *shifted* sinusoidal vector can be written as a weighted sum of two *fixed* {sin, cos} vectors!

Shifting Sinusoids

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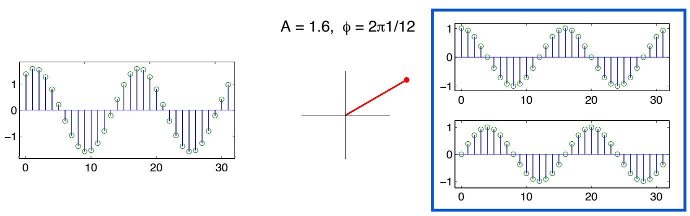
fixed cos/sin vectors:



Any *scaled* and *shifted* sinusoidal vector can be written as a weighted sum of two *fixed* {sin, cos} vectors!

Shifting Sinusoids

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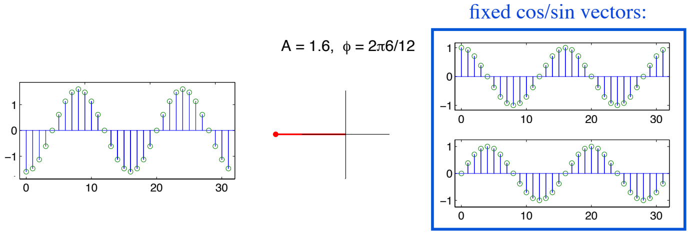
fixed cos/sin vectors:

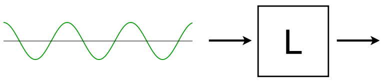


Any *scaled* and *shifted* sinusoidal vector can be written as a weighted sum of two *fixed* {sin, cos} vectors!

LSI response to sinusoids

$x(n) = \cos(\omega n)$ (input)

$$y(n) = \sum_m r(m) \cos(\omega(n-m))$$ (convolution formula)



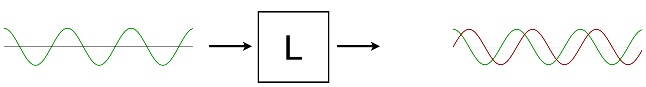
LSI response to sinusoids

$x(n) = \cos(\omega n)$

$$\begin{aligned} y(n) &= \sum_m r(m) \cos(\omega(n-m)) \\ &= \sum_m r(m)\cos(\omega m)\cos(\omega n) + \sum_m r(m)\sin(\omega m)\sin(\omega n) \end{aligned}$$

(trig identity)

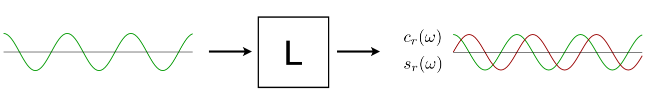
inner product of impulse response with cos/sin, respectively



LSI response to sinusoids

$x(n) = \cos(\omega n)$

$$\begin{aligned} y(n) &= \sum_m r(m) \cos(\omega(n-m)) \\ &= \sum_m r(m)\cos(\omega m)\cos(\omega n) + \sum_m r(m)\sin(\omega m)\sin(\omega n) \\ &= c_r(\omega) \cos(\omega n) + s_r(\omega) \sin(\omega n) \end{aligned}$$

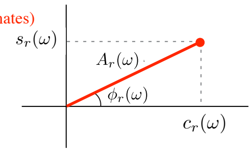


LSI response to sinusoids

$x(n) = \cos(\omega n)$

$$
\begin{aligned}
y(n) &= \sum_m r(m) \cos\left(\omega(n-m)\right) \\
&= \sum_m r(m) \cos(\omega m) \cos(\omega n) \;+\; \sum_m r(m) \sin(\omega m) \sin(\omega n) \\
&= \quad c_r(\omega) \quad \cos(\omega n) \;+\; \quad s_r(\omega) \quad \sin(\omega n) \\
&= A_r(\omega) \cos(\phi_r(\omega)) \cos(\omega n) \;+\; A_r(\omega) \sin(\phi_r(\omega)) \sin(\omega n)
\end{aligned}
$$

(rectangular -> polar coordinates)



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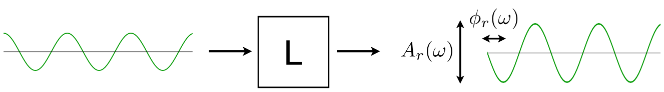
&= \quad c_r(\omega) \quad \cos(\omega n) \;+\; \quad s_r(\omega) \quad \sin(\omega n) \\
&= A_r(\omega) \cos(\phi_r(\omega)) \cos(\omega n) \;+\; A_r(\omega) \sin(\phi_r(\omega)) \sin(\omega n) \\
&= A_r(\omega) \cos(\omega n - \phi_r(\omega))
\end{aligned}
$$

(trig identity, in the opposite direction)



"Sinusoid in, sinusoid out" (with modified amplitude & phase)

LSI response to sinusoids

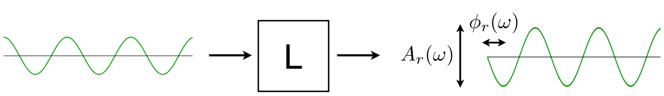
More generally, if input has amplitude A_x and phase ϕ_x ,

$$x(n) = A_x \cos(\omega n - \phi_x)$$

then linearity and shift-invariance tell us that

$$y(n) = A_r(\omega) A_x \cos(\omega n - \phi_x - \phi_r(\omega))$$

amplitudes multiply — phases add



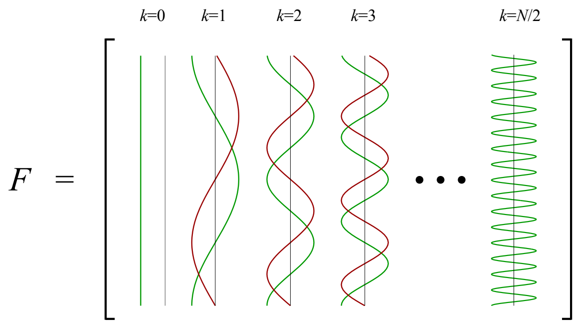
"Sinusoid in, sinusoid out" (with modified amplitude & phase)

The Discrete Fourier transform (DFT)

- Construct an orthogonal matrix of sin/cos pairs, covering different numbers of cycles
- Frequency multiples of $2\pi/N$ radians/sample, (specifically, $2\pi k/N$, for $k = 0, 1, 2, \ldots N/2$)
- For $k = 0$ and $k = N/2$, only need the cosine part (thus, $N/2 + 1$ cosines, and $N/2 - 1$ sines)
- When we apply this matrix to an input vector, think of output as *paired* coordinates
- Common to plot these pairs as amplitude/phase

[details on board...]

Fourier Transform matrix



$\cos\left(\frac{2\pi k}{N} n\right)$ $\sin\left(\frac{2\pi k}{N} n\right)$ (plotted sinusoids are continuous, N=32)

The Fourier family

signal domain

frequency domain	continuous	discrete
continuous	**Fourier transform**	**discrete-time Fourier transform**
discrete	**Fourier series**	**discrete Fourier transform**

(we are here)

The "fast Fourier transform" (FFT) is a computationally efficient implementation of the DFT, requiring *Nlog(N)* operations, compared to the N^2 operations that would be needed for matrix multiplication.

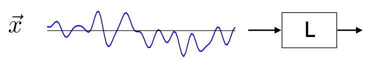
Reminder: LSI response to sinusoids

$$x(n) = \cos(\omega n)$$

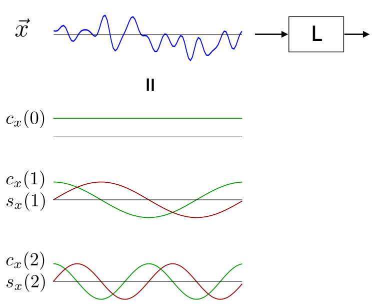
$$\begin{aligned}
y(n) &= \sum_m r(m)\cos\left(\omega(n-m)\right) \\
&= \sum_m r(m)\cos(\omega m)\cos(\omega n) \;+\; \sum_m r(m)\sin(\omega m)\sin(\omega n) \\
&= \quad c_r(\omega) \quad \cos(\omega n) \;+\; \quad s_r(\omega) \quad \sin(\omega n) \\
&= A_r(\omega)\cos(\phi_r(\omega))\cos(\omega n) \;+\; A_r(\omega)\sin(\phi_r(\omega))\sin(\omega n) \\
&= A_r(\omega)\cos(\omega n - \phi_r(\omega))
\end{aligned}$$

These dot products are the Discrete Fourier Transform of the impulse response, *r(m)*!

Fourier & LSI

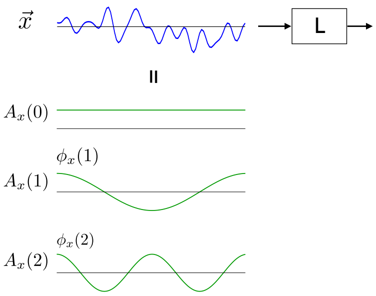


Fourier & LSI



note: only 3 (of many) frequency components shown

Fourier & LSI



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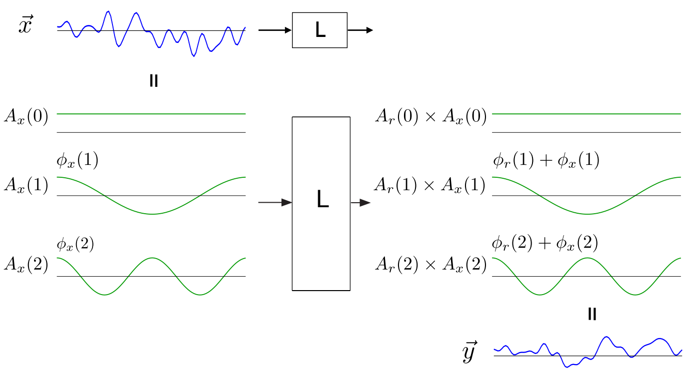
Fourier & LSI



LSI systems are characterized by their *frequency response*, specified by the Fourier Transform of their impulse response

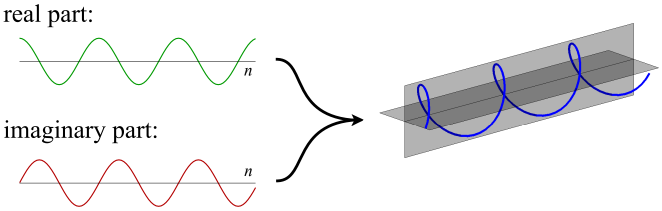
Complex exponentials: "bundling" sine and cosine

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \qquad \text{(Euler's formula)}$$

$$Ae^{i\omega n} = A\cos(\omega n) + iA\sin(\omega n)$$

real part:

imaginary part:



[on board: reminders of addition/multiplication of complex numbers]

Complex exponentials: "bundling" sine and cosine

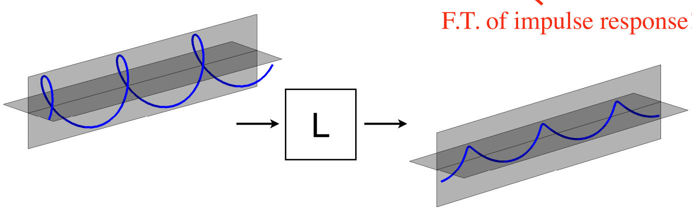
$$e^{i\omega n} \longrightarrow \boxed{L} \longrightarrow A_r(\omega)\, e^{i(\omega n - \phi_r(\omega))} = A_r(\omega)\, e^{-i\phi_r(\omega)}\, e^{i\omega n} = \tilde{r}(\omega)\, e^{i\omega n}$$

F.T. of impulse response!

Complex exponentials: "bundling" sine and cosine

$$e^{i\omega n} \longrightarrow \boxed{L} \longrightarrow A_r(\omega)\, e^{i(\omega n - \phi_r(\omega))} = A_r(\omega)\, e^{-i\phi_r(\omega)}\, e^{i\omega n} = \tilde{r}(\omega)\, e^{i\omega n}$$

F.T. of impulse response!



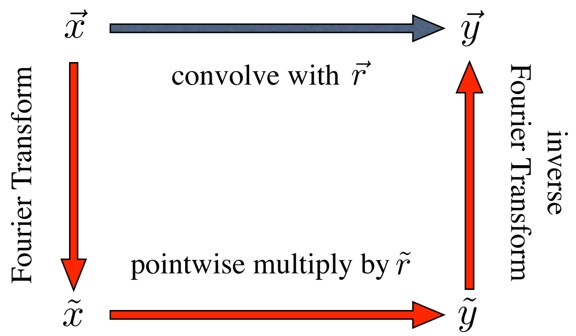
Note: the complex exponentials are *eigenvectors*!

The "convolution theorem"

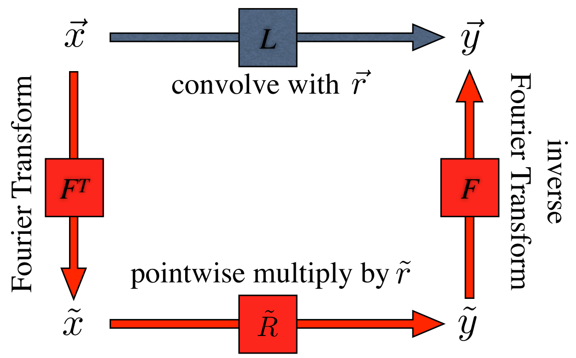
$\vec{x}$ ⟶ $\vec{y}$

convolve with $\vec{r}$

The “convolution theorem”



The “convolution theorem”



$$\vec{y} = L\vec{x} = F\tilde{R}F^T\vec{x} \qquad \Rightarrow \qquad F^T\vec{y} = \tilde{R}F^T\vec{x}$$

(diagonal matrix)

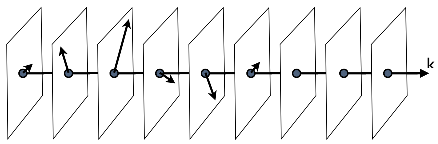
Recap…

- Linear system
  - defined by superposition
  - characterized by a matrix
- Linear Shift-Invariant (LSI) system
  - defined by superposition and shift-invariance
  - characterized by a vector, which can be either:
    - » the impulse response
    - » the frequency response (amplitude and phase). Specifically, the Fourier Transform of the impulse response specifies an amplitude multiplier and a phase shift for each frequency.

Discrete Fourier transform (with complex numbers)

$$\tilde{r}_k = \sum_{n=0}^{N-1} r_n e^{-i\omega_k n} \qquad \text{where} \quad \omega_k = \frac{2\pi k}{N}$$

$$r_n = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{r}_k \; e^{i\omega_k n} \qquad \text{(inverse)}$$



[on board: why minus sign? why 1/N?]

Visualizing the (Discrete) Fourier Transform

- Two conventional choices for frequency axis:
  - Plot frequencies from k=0 to k=N/2
    (in matlab: 1 to N/2-1)
  - Plot frequencies from k=-N/2 to N/2-1
    (in matlab: use fftshift)
- Typically, plot *amplitude* (and possibly *phase*, on a separate graph), instead of the real/ imaginary (cosine/sine) components

Some examples

- constant
- sinusoid (see next slide)
- impulse
- Gaussian - “lowpass”
- DoG (difference of 2 Gaussians) - “bandpass”
- Gabor (Gaussian windowed sinusoid) - “bandpass”

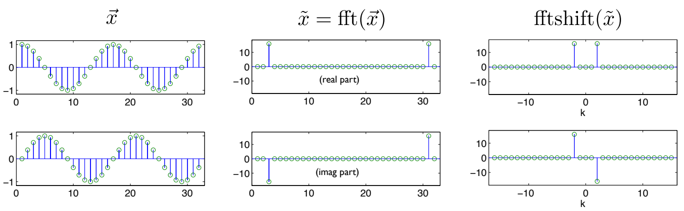
[on board]

$$e^{i\omega n} = \cos(\omega n) + i\sin(\omega n) \qquad e^{-i\omega n} = \cos(\omega n) - i\sin(\omega n)$$

$$\Rightarrow \quad \cos(\omega n) = \frac{1}{2}(e^{i\omega n} + e^{-i\omega n})$$

$$\sin(\omega n) = \frac{-i}{2}(e^{i\omega n} - e^{-i\omega n})$$

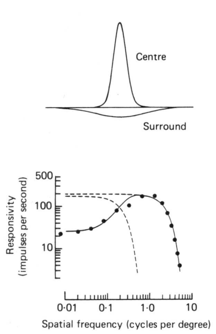
Example for k=2, N=32 (note indexing and amplitudes):

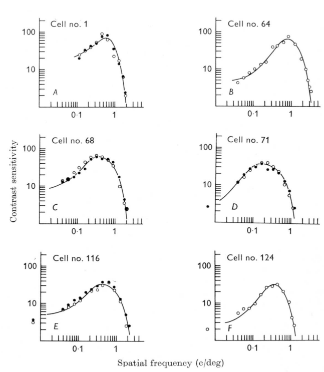


What do we *do* with Fourier Transforms?

- Represent/analyze periodic *signals*
- Analyze/design LSI *systems*. In particular, how do you identify the nullspace?

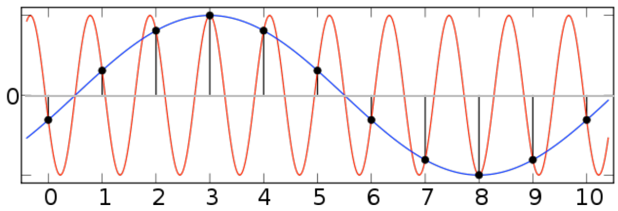
Retinal ganglion cells (1D)





Enroth-Cugell and Robson (1984)

Sampling causes "aliasing"

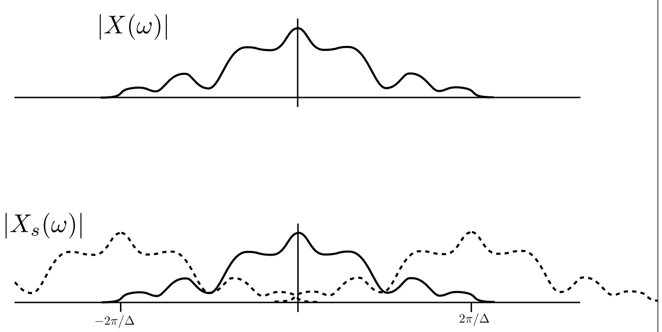


Sampling process is linear, but many-to-one (non-invertible)

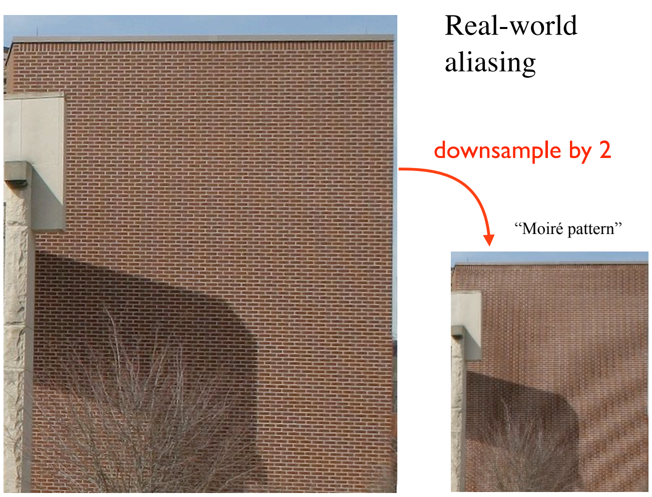
"Aliasing" - one frequency masquerades as another *[on board]*

Given the samples, it is common/natural to assume, or enforce, that they arose from the *lowest* compatible frequency...

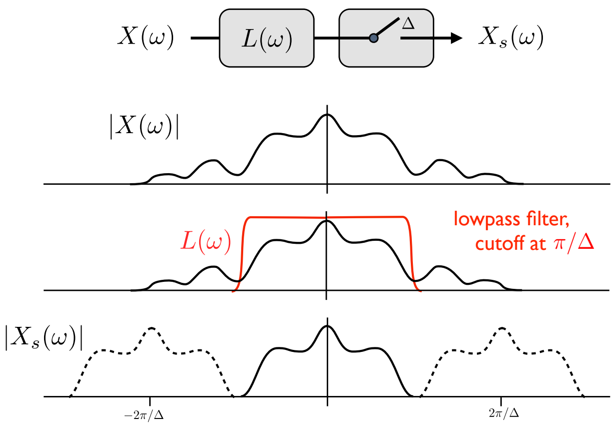
Effect of sampling on the Fourier Transform:
Sum of shifted copies



Real-world aliasing



Pre-filtering to avoid spectral overlap (“aliasing”)



Real-world aliasing