Mathematical Tools
for Neural and Cognitive Science

Fall semester, 2019

Section 1a: Trichromacy
(an extended linear algebra example)

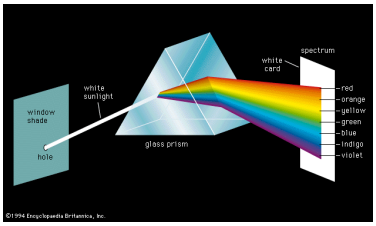
Trichromacy

A spectacular multi-disciplinary scientific story...

- Physics: spectral nature and additivity of light [Newton, and others, 1600's]
- Perception: Trichromatic matching [Grassman, Young, and others, 1850's]
- Mathematical theory: Color matching is explained by a 3-dimensional linear mechanism [late 1800's]
- Engineering: Devices for color reproduction require only three color channels [early 1930's]
- Neurobiology: Trichromacy in humans (and some other primates) arises from 3 cone types [late 1900's]

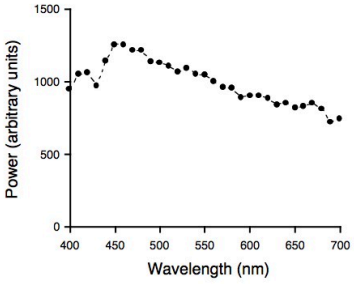
[See Wandell, *Foundations of Vision*, ch2]

Spectral nature of light



[Newton, 1665]

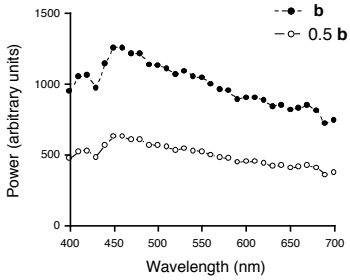
Representing spectra with vectors



$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \bullet \\ \bullet \\ \bullet \\ b_{31} \end{bmatrix}$$

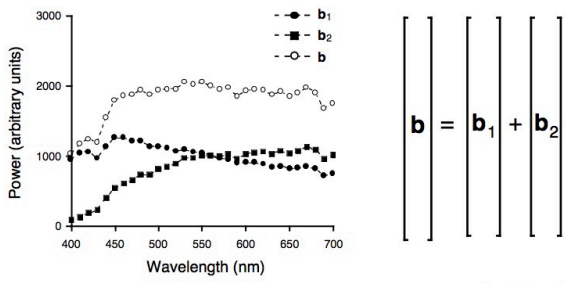
[c/o David Brainard]

Intensity adjustment (scalar multiplication)



[c/o David Brainard]

Additive mixtures



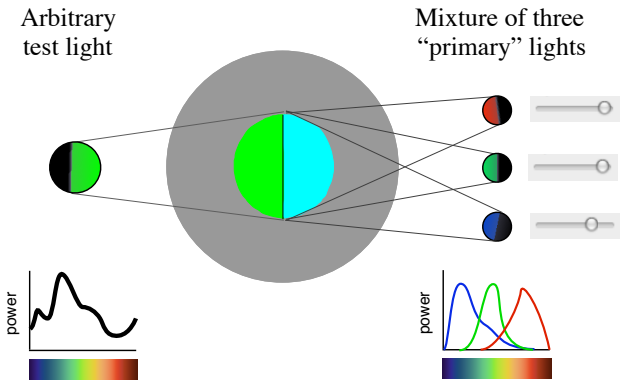
$$\begin{bmatrix} \mathbf{b} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \end{bmatrix} + \begin{bmatrix} \mathbf{b}_2 \end{bmatrix}$$

[c/o David Brainard]

Color perception: what's it for?

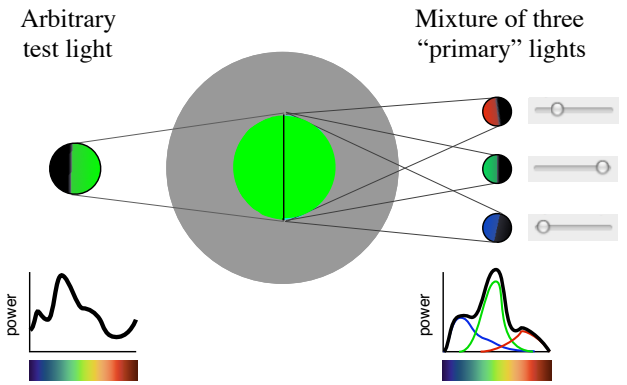
[c/o David Brainard]

Perceptual color matching experiment

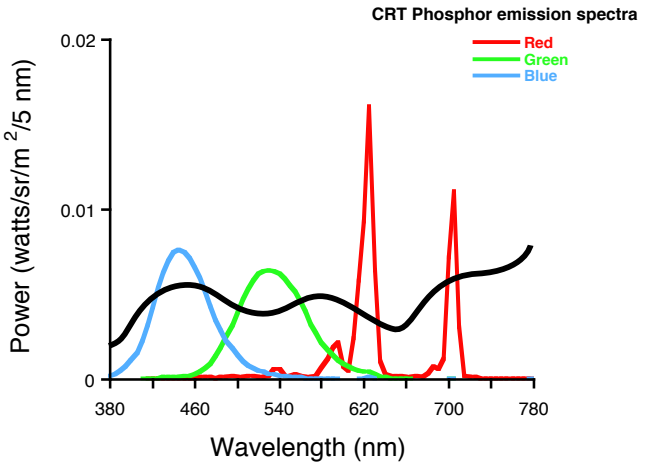


[Young, Helmholtz, Grassman, etc, 1800's]

Perceptual color matching experiment



[Young, Helmholtz, Grassman, etc, 1800's]



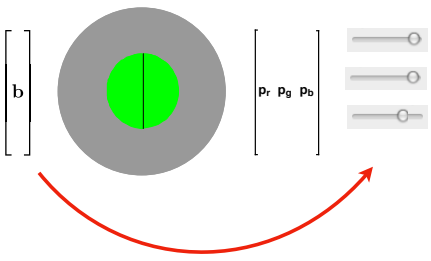
Grassmann's Laws (1853)

1) Any light can be matched with a mixture of 3 primaries
2) Rescaling the light results in a rescaled mixture
3) Adding two lights results in a sum of their mixtures

➡ Color matching can be described by an Nx3 linear system*

* Fine print: i) Normal human observer; ii) photopic intensity levels, not too bright, not too dim; iii) independent primaries; iv) negative primary amplitude = add to test light.

Experimental measurement:

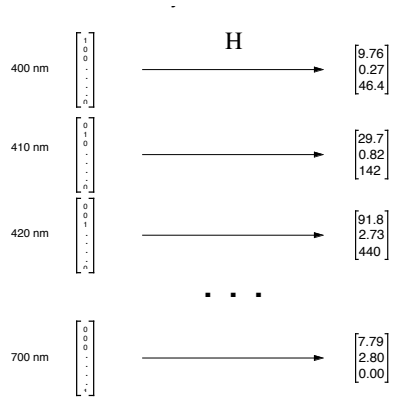


Linear model:

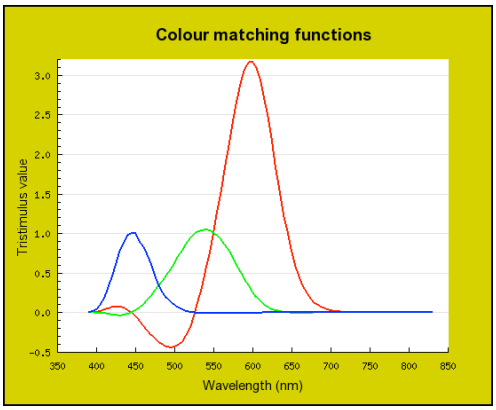
$$\begin{bmatrix} 0.25 \\ 0.75 \\ 0.50 \end{bmatrix} = \begin{bmatrix} \quad \mathbf{H} \quad \end{bmatrix} \begin{bmatrix} b \end{bmatrix}$$

[c/o David Brainard]

Characterization



[c/o David Brainard]



Stiles-Burch Data, Monochromatic RGB Primaries of 645, 526, and 444 nm.

A matrix that mimics a human [Stiles & Burch, 1959]

Implications

- If *P* is an *Nx3* matrix containing the primary spectra, and *H* is an *3xN* "matching matrix" that captures human color matching responses (mapping a spectrum to 3 "knob settings"), then for any light (spectrum) vector $\vec{l}$:

$$\vec{l} \sim PH\vec{l}$$

where $\sim$ means "looks the same as"

- Two lights look the same if (and only if) they produce the same match settings:

$$\vec{l_1} \sim \vec{l_2} \quad \Leftrightarrow \quad H\vec{l_1} = H\vec{l_2}$$

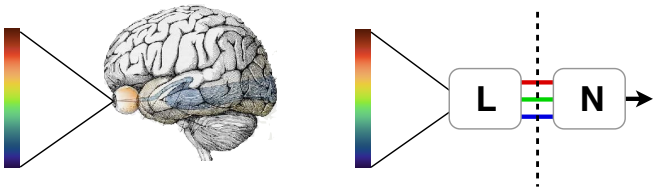
Useful calculations

- Scientific: given results of one matching experiment, predict the results of another one (with different primaries).
- Practical: calibrate a display device, so as to generate mixtures of three colors that match the appearance of any desired real-world spectrum.

[derive on board]

In summary:

Theory: the visual system projects the wavelength spectra of light onto a 3-dimensional subspace



- Predicts/explains perceptual "metamers" - lights that appear identical, but have physically distinct wavelength spectra (mid/late 1800's).
- Codified in CIE standards for color representation in 1931.
- Underlying mechanism (cones) verified in 1987!

The underlying mechanism...