Mathematical Tools
for Neural and Cognitive Science

Fall semester, 2019

Section 1: Linear Algebra

Linear Algebra

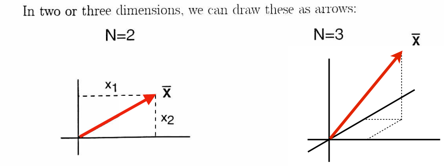
"Linear algebra has become as basic and as applicable as calculus, and fortunately it is easier"

- Gilbert Strang, *Linear Algebra and its Applications*

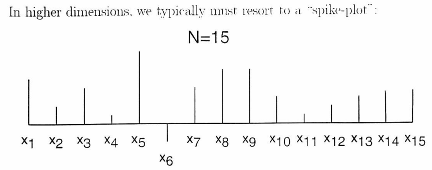
Vectors

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_N \end{pmatrix}$$

In two or three dimensions, we can draw these as arrows:



In higher dimensions, we typically must resort to a "spike-plot":

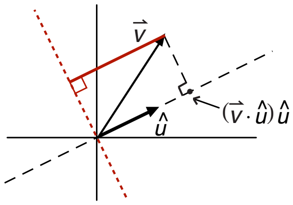


Vector operations

- scalar multiplication
- addition, vector spaces
- length, unit vectors
- inner product (a.k.a. “dot” product)
  - properties: commutative, distributive
  - geometry: cosines, orthogonality test

[on board: geometry]

Inner product with a unit vector



- projection
- distance to line
- change of coordinates

[on board: geometry]

Vectors as “operators”

- “averager”
- “windowed averager”
- “gaussian averager”
- “local differencer”
- “component selector”

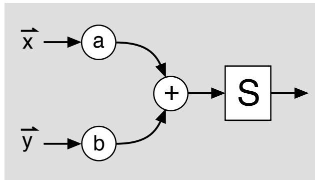
[on board]

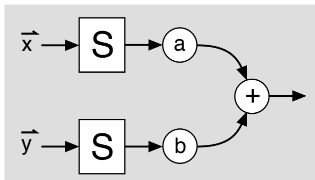
Linear System

S is a linear system if (and only if) it obeys the principle of superposition:

$$S(a\vec{x} + b\vec{y}) = aS(\vec{x}) + bS(\vec{y})$$

For *any* input vectors $\{\vec{x}, \vec{y}\}$, and *any* scalars $\{a, b\}$, the two diagrams at the right must produce the same response:

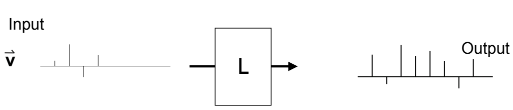




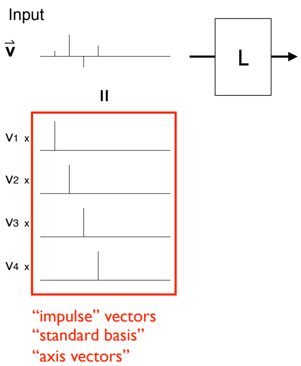
Linear Systems

- Very well understood (150+ years of effort)
- Excellent design/characterization toolbox
- An idealization (they do not exist!)
- Useful nevertheless:
  - conceptualize fundamental issues
  - provide baseline performance
  - good starting point for more complex models

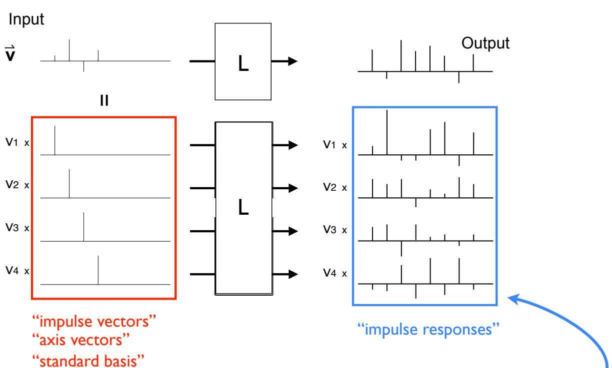
Implications of Linearity



Implications of Linearity



Implications of Linearity



Response to *any* input can be predicted from responses to impulses
This defines the operation of *matrix multiplication*

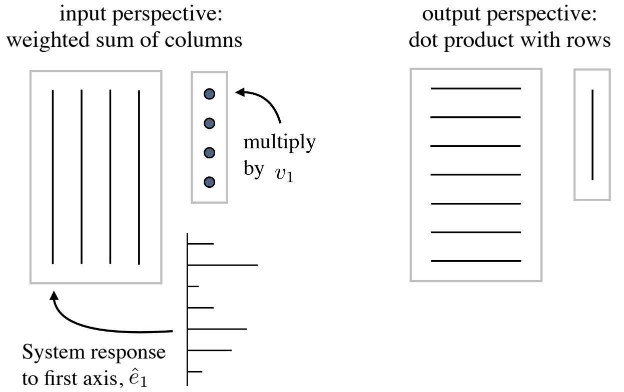
Matrix multiplication

- Two interpretations of $M\vec{v}$ (see next slide):
  - input perspective: weighted sum of columns (from diagrams on previous slides)
  - output perspective: inner product with rows
- distributive property (directly from linearity!)
- associative property: cascade of two linear systems defines the product of two matrices
- transpose A^T, symmetric matrices ($A = A^T$)
- generally *not* commutative ($AB \neq BA$), but note that $(AB)^T = B^T A^T$
- Vectors: Inner products, Outer products

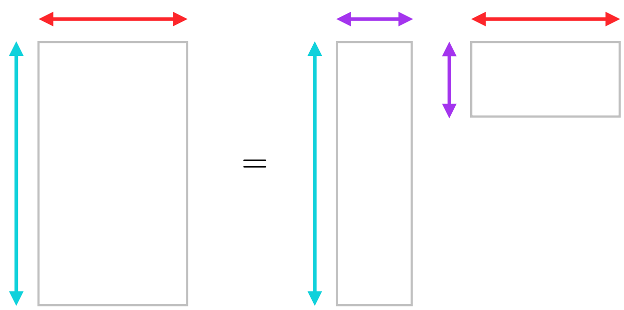
[details on board]

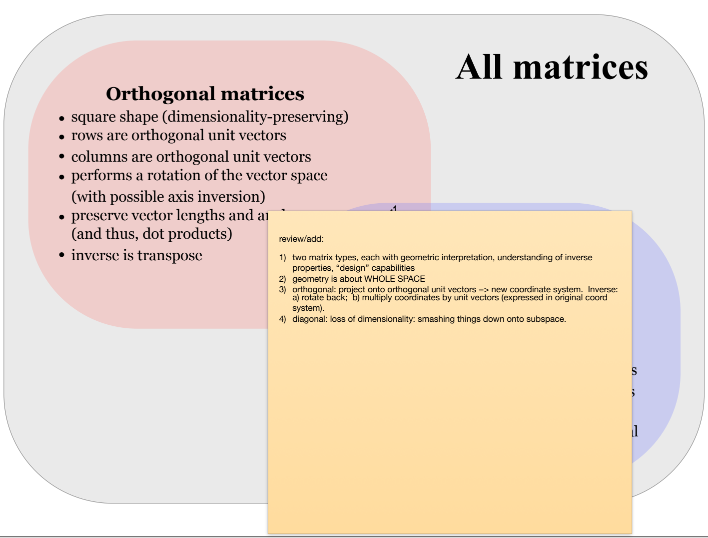
Matrix multiplication

Two interpretations of $M\vec{v}$:



Matrix multiplication: dimensional consistency





Singular Value Decomposition (SVD)

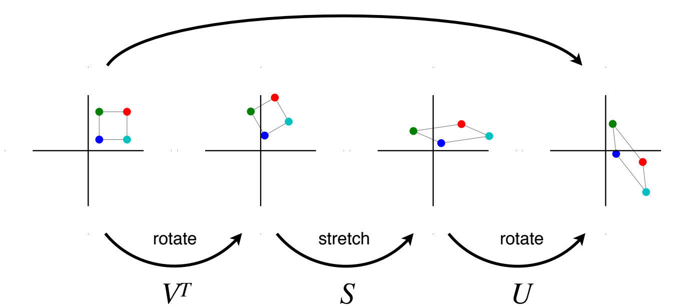
- can express *any* matrix as $M = U\,S\,V^T$
  "rotate, stretch, rotate"
  - columns of V are basis for input coordinate system
  - columns of U are basis for output coordinate system
  - S rescales axes, and determines what "gets through"
- interpretation: sum of "outer products"
- non-uniqueness? permutations, sign flips
- nullspace and rangespace
- inverse and pseudo-inverse

[details on board]

SVD geometry (in 2D)

Consider applying M to four vectors (colored points)

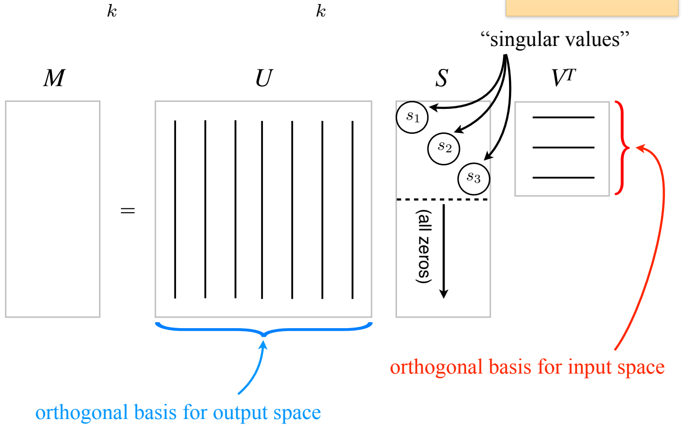
$$M = U\,S\,V^T$$

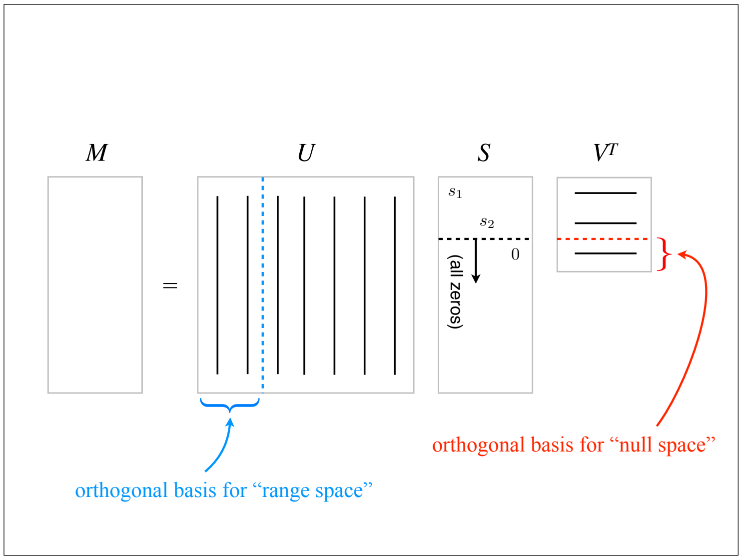


(note order of transformations!)

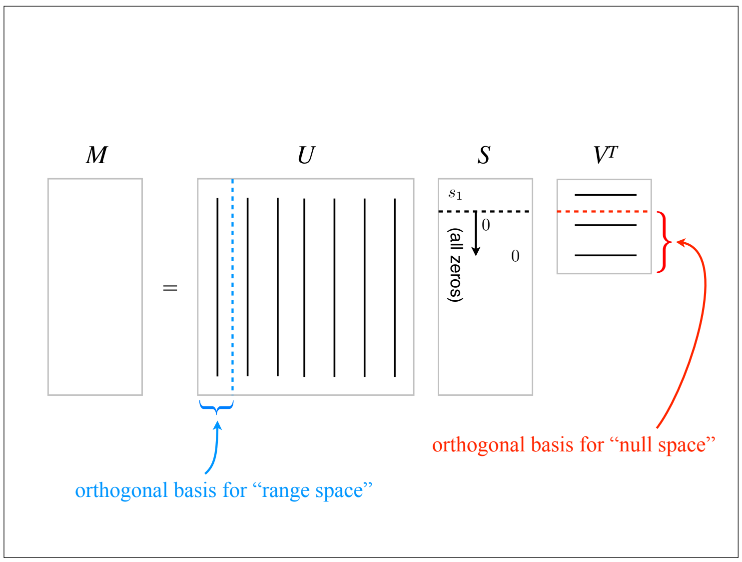
$$M\vec{w} = \sum_k s_k \left(\vec{v}_k^T \vec{w}\right) \vec{u}_k = \sum_k s_k \left(\vec{u}_k \vec{v}_k^T\right) \vec{w}$$

Use hats instead of vectors for u,v. make input \vec{x}





M
U
S
V^T
=
s_1
s_2
0
(all zeros)
orthogonal basis for "range space"
orthogonal basis for "null space"



M
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