PSYCH-GA.2211/NEURL-GA.2201 – Fall 2016
Mathematical Tools for Cognitive and Neural Science

Homework 3

Due: 8 Nov 2016
(late homeworks penalized 10% per day)

See the course web site for submission details. Don't wait until the day before the due date... start *now*!

1. **Linear shift-invariant (time-invariant) systems**. Written exercises: Oppenheim & Schafer, problems 2.35 and 2.36 [see attached pages]. Note: please *explain* your answers! Note: This problem index values n that represent time, and can be positive or negative. The $\delta[n]$ indicates the standard "impulse" – a vector that is zero everywhere except at index $n = 0$, where it is one.

2. **LSI system characterization.** You are experimenting with three unknown systems, embodied in compiled matlab functions `unknownSystemX.p`, (with X=1,2,3), that each take an input column vector of length $N = 64$. The response of each is a column vector (of the same length). Your task is to examine them to see if they behave like they're linear and/or shift-invariant with circular (periodic) boundary-handling. For each system,

   (a) "Kick the tires" by measuring the response to an impulse in the first position of an input vector. Check that this impulse response is shift-invariant by comparing to the response to an impulse in a few later positions. Also check that the response to a sum of two of these impulses is equal to the sum of their individual responses.

   (b) If the previous tests were positive, examine the response of the system to sinusoids with frequencies $\{2\pi/N, 4\pi/N, 6\pi/N, 12\pi/N\}$, and random phases, and check whether the outputs are sinusoids of the same frequency (i.e., verify that the output vector lies completely in the subspace containing all the sinusoids of that frequency).

   (c) If the previous tests were positive, verify that the change in amplitude and phase from input to output is predicted by the amplitude (`abs`) and phase (`angle`) of the corresponding terms of the Fourier transform of the impulse response. If not, explain which property (linearity, or shift-invariance, or both) seems to be missing from the system.

3. **Convolution in matlab**. Create a vector of length 3, `r = [3 2 1]`, and suppose this is the finite-length impulse response of a linear shift-invariant system. Because it is LSI, the response of this system to any input vector `in` can be computed as a convolution.

   (a) Compute responses to the eight 8-dimensional impulse vectors, using MATLAB's `conv` function: `out = conv(in, r)`. How do these compare to what you'd expect from the convolution formula given in class, $y(n) = \sum_k r(n-k)x(k)$? Specifically, (a) compute the matrix that represents the linear system. What is the size, and organization of this matrix?

   (b) How does MATLAB's `conv` function handle boundaries?

(c) Using `conv`, compute the response to an input vector of length 48 containing a single-cycle cosine. Is this a single-cycle sinusoid? Why or why not? If not, what modification is necessary to the `conv` function to ensure that it will behave according to the "sine-in, sine-out" behavior we expect of LSI systems?

4. **Bandpass Difference-of-Gaussians (DoG) filter.**

   (a) Create a one-dimensional linear filter that is a difference of two Gaussians, $e^{-\frac{n^2}{2\sigma^2}}$, (each normalized to sum to 1), and with standard deviations $\sigma = 1.5$ and $\sigma = 3.5$ samples. The filter should contain 15 samples, with both Gaussians centered on the middle (8th) sample.

   (b) Plot the amplitude of the Fourier transform of this filter, sampled at 32 locations (MATLAB's `fft` function takes an optional additional argument). What kind of filter is this? If you were to convolve this filter with sinusoids of different frequencies, which of them (approximately) would produce a response with the largest amplitude? Which of them would produce responses with about 25% of this maximal amplitude?

   (c) Create three unit-amplitude 32-sample sinusoidal signals at the three frequencies (low, mid, high) that you found in part (b). Convolve the filter with each, and verify that the amplitude of the response is approximately consistent with the answers you gave in part (b). (hint: either project the response onto sine and cosine of the appropriate frequency, or compute the DFT of the response and measure ampitude at the appropriate frequency).

5. **Sampling and aliasing.** Load the file `myMeasurements.mat` into matlab. It contains a vector, named `sig`, containing a set of voltage values measured from an EEG electrode, at 128 equi-spaced moments in time. Plot `sig` against the time values (variable `time`), showing both the values, and the lines connecting the values (use the flag 'ko-' in matlab's plot command).

   - You don't want to store all those values, and decide instead to keep *every fourth value*. Compute a new signal that contains these values. Plot it, against the corresponding entries of the `time` vector (if you prefer, you can plot it on top of the original data, using matlab's `hold` function, and with plot flag 'r*-'). How does this reduced version of the data look, compared to the original? Does it provide a good summary of the original measurements?
   - To understand what's going on, look at the signal in the frequency domain. First plot the magnitude (amplitude) of the Fourier transform of the original signal, over the range $[-N/2, (N/2) - 1]$ (use `fftshift`). Now plot the same thing for the subsampled signal. Explain the relationship between the two plots. In particular, what has happened to the frequency components of the original signal? How does this transformation explain the appearance of the subsampled signal? [Hint: remember that the DFT is periodic. It might help to plot four periods of the DFT of the subsampled signal, which will be the same size as the DFT of the original signal].

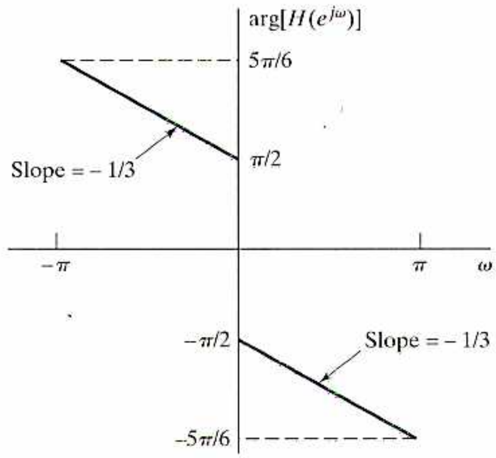

Figure P2.33-1

2.34. The input–output pair shown in Figure P2.34-1 is given for a stable LTI system.

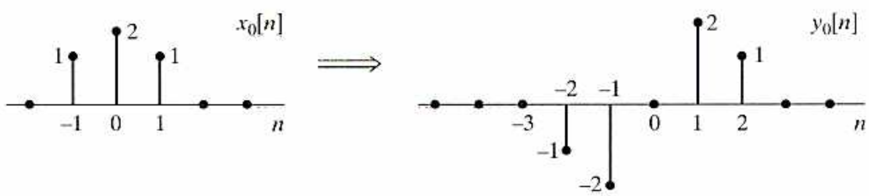

Figure P2.34-1

(a) Determine the response to the input $x_1[n]$ in Figure P2.34-2.

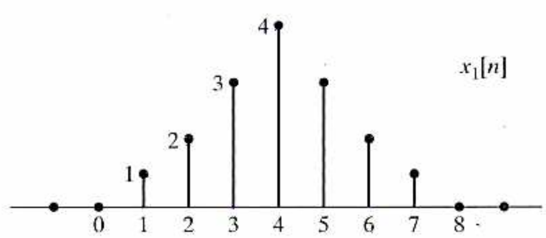

Figure P2.34-2

(b) Determine the impulse response of the system.

Advanced Problems

2.35. The system T in Figure P2.35-1 is known to be *time invariant*. When the inputs to the system are $x_1[n]$, $x_2[n]$, and $x_3[n]$, the responses of the system are $y_1[n]$, $y_2[n]$, and $y_3[n]$, as shown.

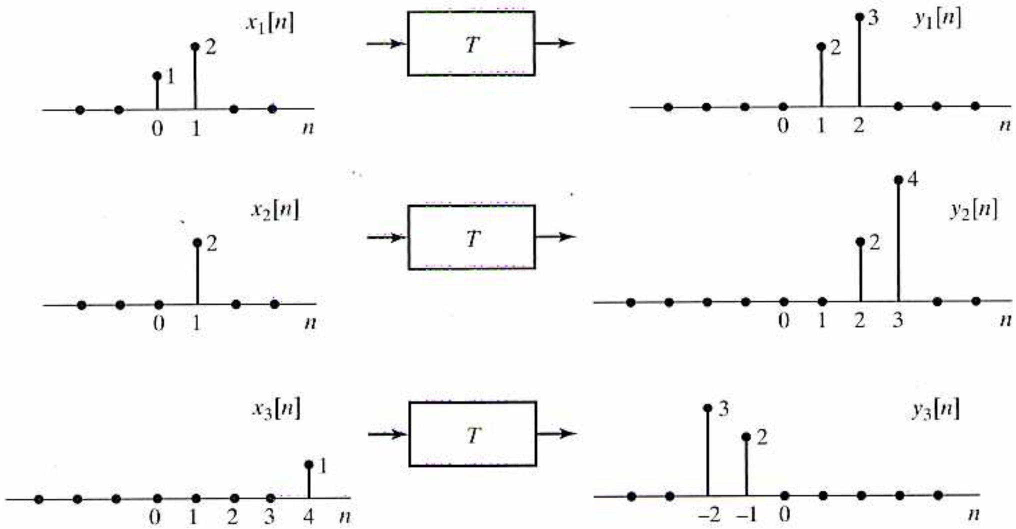

Figure P2.35-1

(a) Determine whether the system T could be linear.
(b) If the input $x[n]$ to the system T is $\delta[n]$, what is the system response $y[n]$?
(c) What are all possible inputs $x[n]$ for which the response of the system T can be determined from the given information alone?

2.36. The system L in Figure P2.36-1 is known to be *linear*. Shown are three output signals $y_1[n]$, $y_2[n]$, and $y_3[n]$ in response to the input signals $x_1[n]$, $x_2[n]$, and $x_3[n]$, respectively.

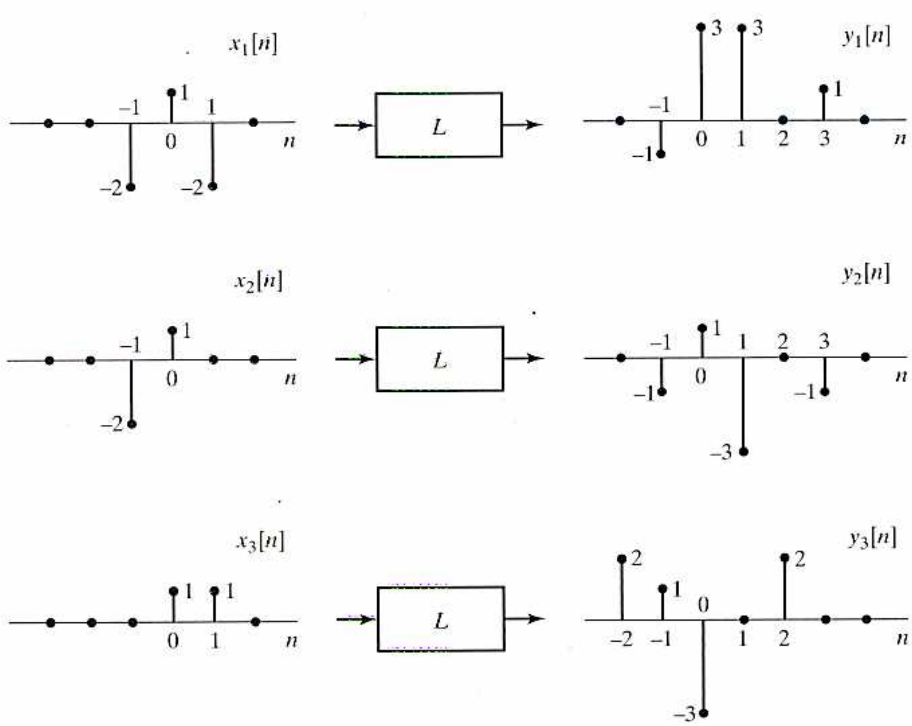

Figure P2.36-1

(a) Determine whether the system L could be time invariant.
(b) If the input $x[n]$ to the system L is $\delta[n]$, what is the system response $y[n]$?

2.37. Consider a discrete-time linear time-invariant system with impulse response $h[n]$. If the input $x[n]$ is a periodic sequence with period N (i.e., if $x[n] = x[n + N]$), show that the output $y[n]$ is also a periodic sequence with period N.

2.38. In Section 2.5, we stated that the solution to the homogeneous difference equation

$$\sum_{k=0}^{N} a_k y_h[n-k] = 0 \tag{P2.38-1}$$

is of the form

$$y_h[n] = \sum_{m=1}^{N} A_m z_m^n, \tag{P2.38-2}$$

with the A_m's arbitrary and the z_m's the N roots of the polynomial

$$\sum_{k=0}^{N} a_k z^{-k} = 0; \tag{P2.38-3}$$

i.e.,

$$\sum_{k=0}^{N} a_k z^{-k} = \prod_{m=1}^{N} (1 - z_m z^{-1}). \tag{P2.38-4}$$

(a) Determine the general form of the homogeneous solution to the difference equation

$$y[n] - \tfrac{3}{4} y[n-1] + \tfrac{1}{8} y[n-2] = 2x[n-1]. \tag{P2.38-5}$$

(b) Determine the coefficients A_m in the homogeneous solution if $y[-1] = 1$ and $y[0] = 0$.
(c) Now consider the difference equation

$$y[n] - y[n-1] + \tfrac{1}{4} y[n-2] = 2y[n-1]. \tag{P2.38-6}$$

If the homogeneous solution contains only terms of the form of Eq. (P2.38-2), show that the initial conditions $y[-1] = 1$ and $y[0] = 0$ cannot be satisfied.

(d) If Eq. (P2.38-3) has two roots that are identical, then, in place of Eq. (P2.38-2), $y_h[n]$ will take the form

$$y_h[n] = \sum_{m=1}^{N-1} A_m z_m^n + n B_1 z_1^n, \tag{P2.38-7}$$

where we have assumed that the double root is z_1. Using Eq. (P2.38-7), determine the general form of $y_h[n]$ for Eq. (P2.38-6). Verify explicitly that your answer satisfies Eq. (P2.38-6) with $x[n] = 0$.

(e) Determine the coefficients A_1 and B_1 in the homogeneous solution obtained in Part (d) if $y[-1] = 1$ and $y[0] = 0$.

2.39. Consider a system with input $x[n]$ and output $y[n]$. The input–output relation for the system is defined by the following two properties:

1. $y[n] - ay[n-1] = x[n]$,
2. $y[0] = 1$.

(a) Determine whether the system is time invariant.
(b) Determine whether the system is linear.