Mathematical Tools
for Neural and Cognitive Science

Fall semester, 2016

Section 1: Linear Algebra

Linear Algebra

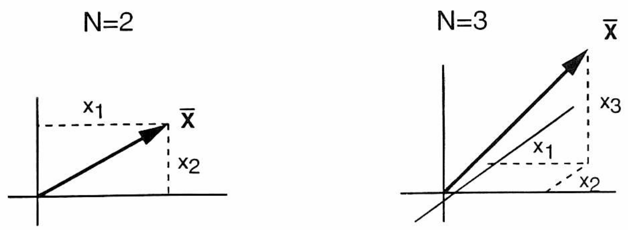
"Linear algebra has become as basic and as applicable as calculus, and fortunately it is easier"

- Gilbert Strang, *Linear Algebra and its Applications*

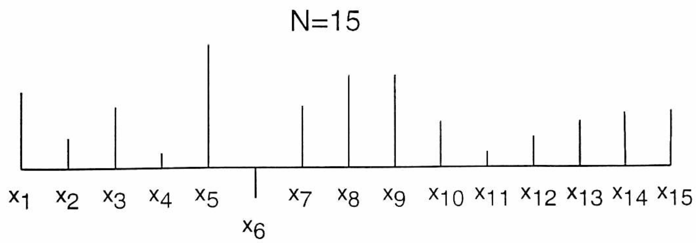
Vectors

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_N \end{pmatrix}$$

In two or three dimensions, we can draw these as arrows:



In higher dimensions, we typically must resort to a "spike-plot":



Vector operations

- scalar multiplication
- addition, vector spaces
- length, unit vectors
- inner product (a.k.a. "dot" product)
  - properties: commutative, distributive
  - geometry: cosines, orthogonality test

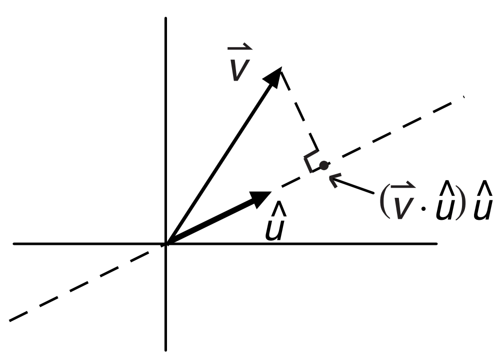
[on board: geometry]

Vectors as "operators"

- "averager"
- "windowed averager"
- "gaussian averager"
- "local differencer"
- "component selector"

[answers on board]

Inner product with a unit vector



- projection
- distance
- change of coordinates

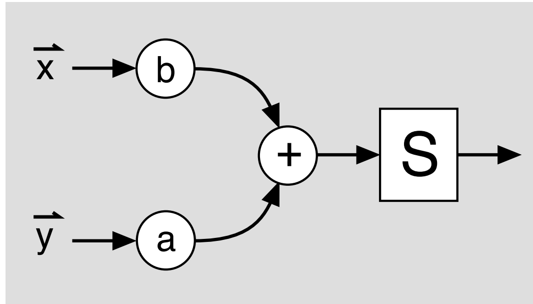
[on board: geometry]

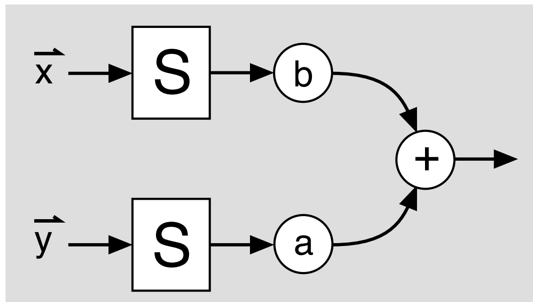
Linear System

S is a linear system if (and only if) it obeys the principal of superposition:

$$S(a\vec{x} + b\vec{y}) = aS(\vec{x}) + bS(\vec{y})$$

For *any* input vectors $\{\vec{x}, \vec{y}\}$, and *any* scalars $\{a, b\}$, the two diagrams at the right must produce the same response:

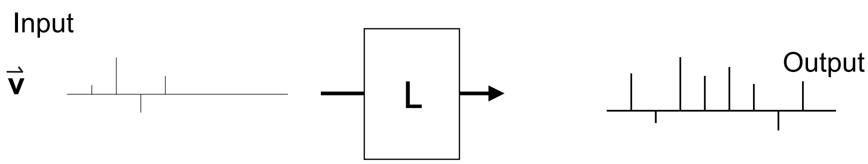




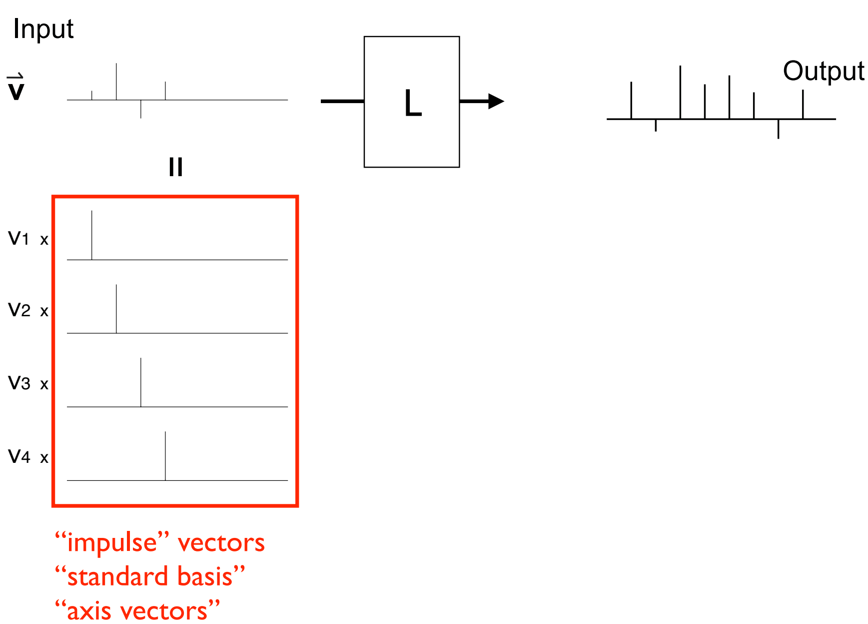
Linear Systems

- Very well understood (150+ years of effort)
- Excellent design/characterization toolbox
- An idealization (they do not exist!)
- Useful nevertheless:
  - conceptualize fundamental issues
  - provide baseline performance
  - good starting point for more complex models

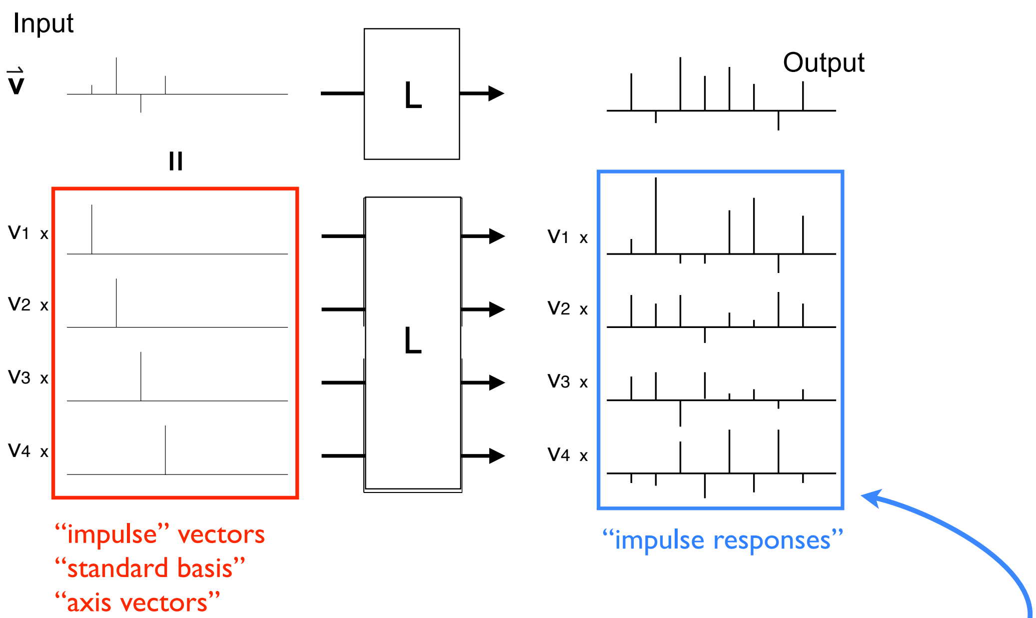
Implications of Linearity



Implications of Linearity



Implications of Linearity



Response to *any* input can be predicted from responses to impulses
This defines the operation of *matrix multiplication*

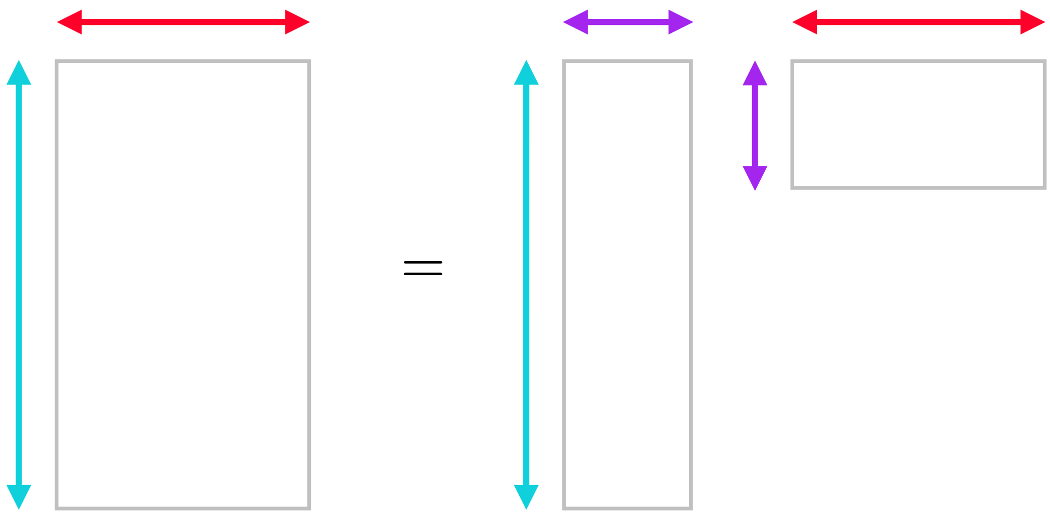
Matrix multiplication

- input perspective: weighted sum of columns

  (from diagram on previous slide)

- output perspective: inner product with rows
- distributive property (directly from linearity)
- associative property - cascade of two linear systems defines the product of two matrices
- generally *not* commutative ($AB \neq BA$), but note that $(AB)^T = B^T A^T$

Matrix multiplication: dimensional consistency



Matrix types

All matrices

Orthogonal matrices

- square shape (dimensionality-preserving)
- rows are orthogonal unit vectors
- columns are orthogonal unit vectors
- performs a rotation of the vector space (with possible axis inversion)
- preserve vector lengths and angles (and thus, dot products)
- inverse is transpose

Identity matrix

Diagonal matrices

- arbitrary rectangular shape
- all off-diagonal entries are zero
- squeeze/stretch along standard axes
- if non-square: creates/discards axes
- inverse is diagonal, with inverse of non-zero diagonal entries of original

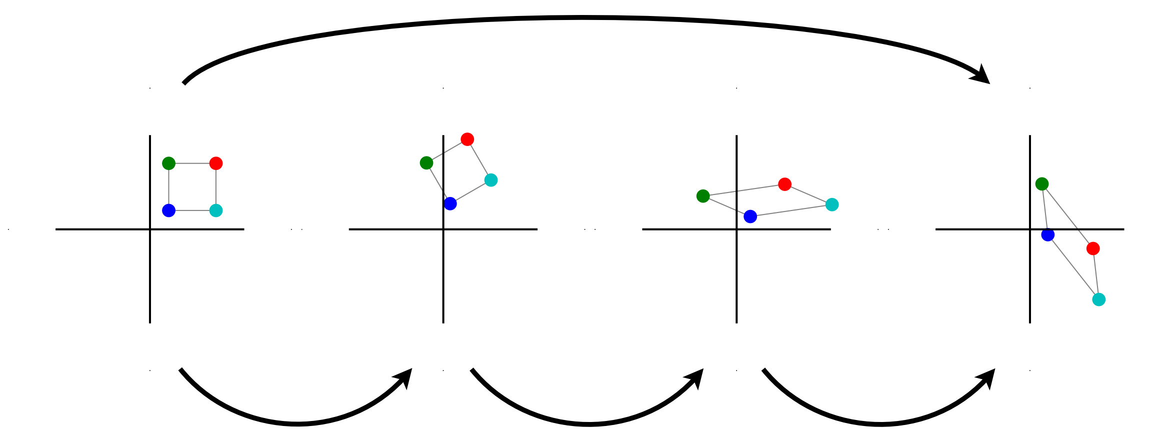
Singular Value Decomposition (SVD)

- $M = U\ S\ V^T$,   "rotate, stretch, rotate"
- V is input coordinate system (U, output)
- interpretation as sum of outer products
- non-uniqueness (permutations, sign flips)
- nullspace and rangespace
- inverse and pseudo-inverse

[details on board]

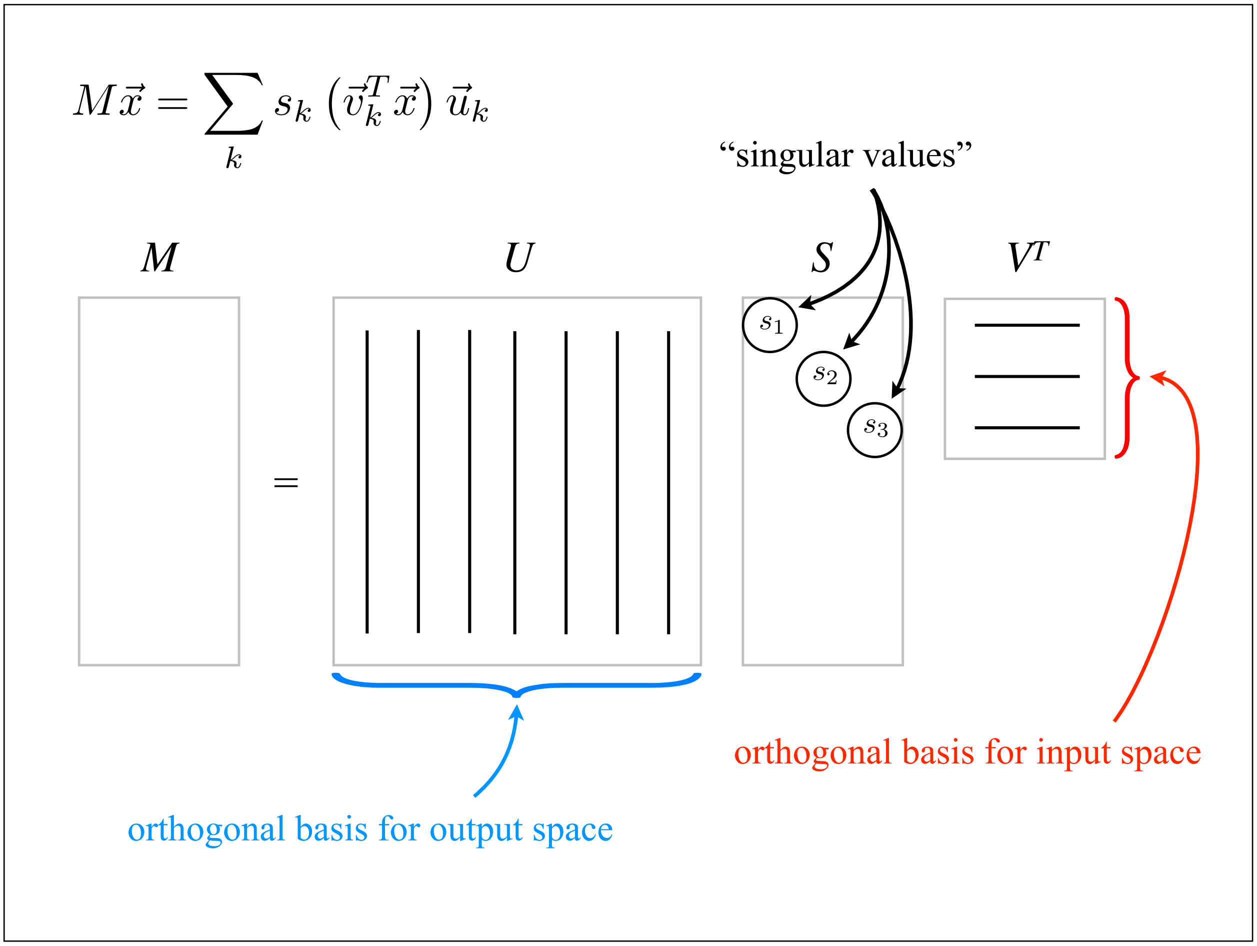
SVD geometry

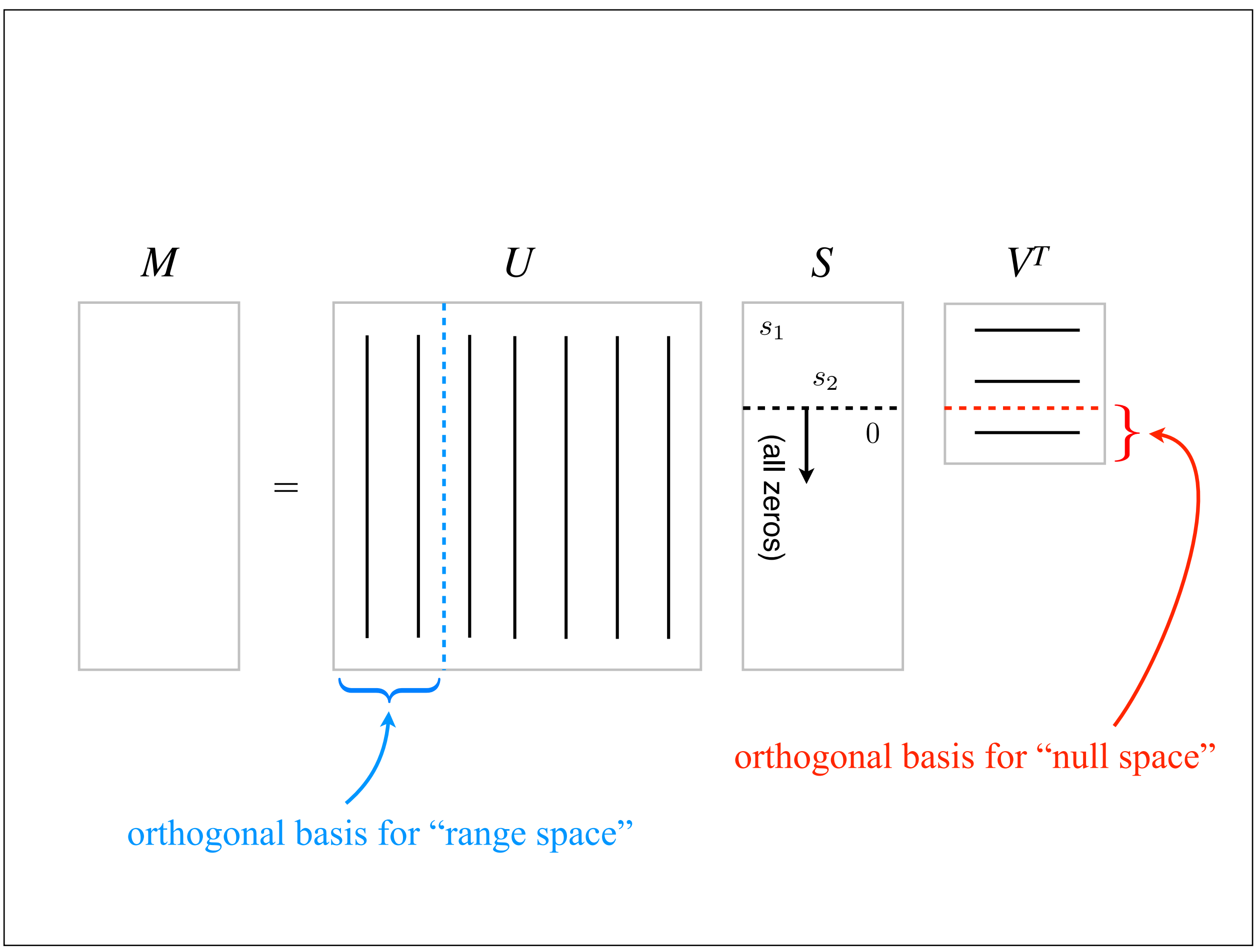
$M = U\ S\ V^T$

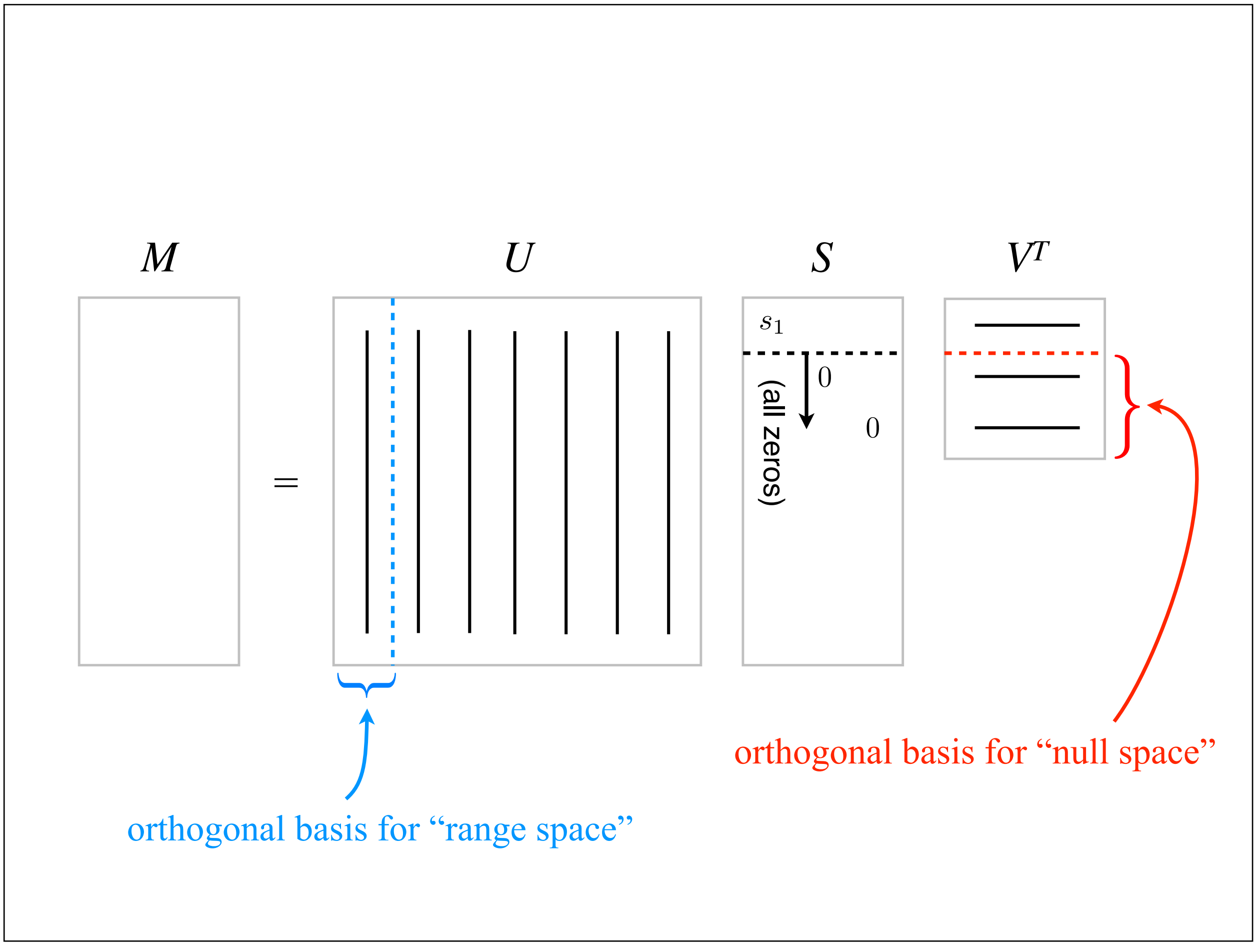


V^T S U

(note order of transformations)


$M\vec{x} = \sum_k s_k \left(\vec{v}_k^T \vec{x}\right) \vec{u}_k$
"singular values"
M
U
S
V^T
s_1
s_2
s_3
=
orthogonal basis for input space
orthogonal basis for output space


M
U
S
V^T
s_1
s_2
0
(all zeros)
=
orthogonal basis for "null space"
orthogonal basis for "range space"



M
=
U
S
V^T
s_1
0
(all zeros)
0
orthogonal basis for "range space"
orthogonal basis for "null space"