PSYCH-GA.2211/NEURL-GA.2201 – Fall 2014
Mathematical Tools for Cognitive and Neural Science

Homework 3

Due: 5 Nov 2014
(late homeworks penalized 10% per day)

Save the solutions to each numbered problem as sections of a file called `hw3.m` in a folder called `hw3_Lastname`, along with additional files containing any functions you create. Send a zipped copy of this folder as an attachment in an email message with these attributes:

```
To: catherio@nyu.edu, asr443@nyu.edu
Subject: Math Tools HW3
```

Don't wait until the day before the due date... start *now*!

1. **Principal components.** Load the file `PCA.mat` into your MATLAB environment. You'll find a matrix M, whose rows contain data in the form of 4-vectors. Each row gives the firing rate of a single neuron under four different stimulus conditions. We cannot visualize the data in this form, but would like to know how the neurons as a population are encoding these four stimuli. For example, we'd like to know how well we could distinguish between these stimuli by observing these neural responses.

   (a) First, modify the matrix M, re-centering the data around zero by subtracting the mean of the rows (a 4-vector) from all rows. Compute the principal components of the re-centered data in two ways: using `svd` and using `eig`, verifying that these give the same answer. Also compute the four associated eigenvalues (or equivalently, the squared singular values) associated with each vector. Do the data points live close to a subspace of dimensionality less than four?

   Now compute the eigenvalues, λ_k, of the covariance matrix $M^T M$ (or, alternatively, the squared singular values of M), and plot them as a function of k, for $k = 1, 2, 3, 4$. Do the data points live close to a subspace of dimensionality less than four?

   (b) Look at the axes of the subspace where the neural responses are varying most (i.e., the eigenvectors corresponding to the largest eigenvalues). How would you describe these? Which stimuli can be easily distinguished by looking at the neural responses, and which cannot be?

   (c) Project the re-centered data in M onto the first principal component (i.e., compute the inner product of the data vectors with the eigenvector corresponding to the maximal eigenvalue). Plot a histogram (using `hist`) of these values. Show that the sum of squares of these values is equivalent to λ_1. What proportion of the total variability of the data (sum of squared norms of all data vectors) does this component account for?

   (d) Show a scatter plot of the data projected onto the first two principal components (that is, plot the inner product of the data with the first component versus the inner product with the second component). Use `plot`, requesting circular plot symbols and no connecting lines. Use `axis('equal')` to set the two axes to use equal scales. Show that the sum of the squared lengths of these projected vectors is equal to $\lambda_1 + \lambda_2$. What proportion of the total variability of the data do these two components account for?

2. **Linear shift-invariant (time-invariant) systems**. Written exercises: Oppenheim & Schafer, problems 2.35 and 2.36 [see attached pages]. Note: $\delta[n]$ indicates a signal that contains a single impulse at location $n = 0$.

3. **LSI system characterization.** You are experimenting with three unknown systems, embodied in compiled matlab functions `unknownSys1.p`, `unknownSys2.p`, and `unknownSys3.p` that each take an input column vector of length $N = 48$. The response of each is a column vector (of the same length). Your task is to examine them to see if they behave like they're linear and/or shift-invariant with circular boundary-handling. For each system,

   (a) "Kick the tires" by measuring the response to an impulse in the first position of a vector of length $N = 48$. Check that this impulse response is shift-invariant by comparing to the response to an impulse in a few later positions. Check that the response to a sum of two of these impulses is equal to the sum of their individual responses.

   (b) If the previous tests were positive, examine the response of the system to sinusoids with frequencies $\{2\pi/N, 4\pi/N, 6\pi/N, 12\pi/N\}$, and random phases, and check whether the outputs are sinusoids of the same frequency (i.e., verify that the output vector lies completely in the subspace containing all the sinusoids of that frequency).

   (c) If the previous tests were positive, verify that the change in amplitude and phase of the output sinusoids is predicted by the amplitude (`abs`) and phase (`angle`) of the appropriate term of the Fourier transform of the impulse response gathered in the first part.

4. **Convolution in matlab**. Create a random vector of length 3, `r = rand(3,1)`, and suppose this is finite-length impulse response of a linear shift-invariant system. Because it is LSI, the response of this system to any input vector `in` can be computed as a convolution.

   (a) For input vectors of length 8, compute the matrix that represents the linear system. What is the size, and organization of this matrix?

   (b) How does MATLAB's `conv` function handle boundaries?

   (c) Using `conv`, compute the response to an input vector of length 48 containing a single-cycle cosine. Is this a single-cycle sinusoid? Why or why not? If not, what modification is necessary to the `conv` function to ensure that it will behave according to the "sine-in, sine-out" behavior expected of LSI systems?

5. **Bandpass Difference-of-Gaussians (DoG) filter.**

   (a) Create a one-dimensional linear filter that is a difference of two Gaussians (each normalized to sum to 1), and with standard deviations 1.5 and 3.5 samples. The filter should contain 15 samples, with both Gaussians centered on the middle (8th) sample.

   (b) Plot the amplitude of the Fourier transform of this filter, sampled at 32 locations (MATLAB's `fft` function takes an optional additional argument). What kind of filter is this? Estimate (by eye) the frequency at which the filter will have maximal response. Estimate (by eye) a lower and a higher frequency at which the response is about 25% of the peak.

   (c) Create three unit-amplitude 32-sample sinusoidal signals at the three frequencies (low, mid, high) that you found in part (b). Convolve the filter with each, and verify that the amplitude of the response is consistent with the answers you gave in part (b). (hint: either project the response onto sine and cosine of the appropriate frequency, or compute the DFT of the response and measure ampitude at the appropriate frequency).

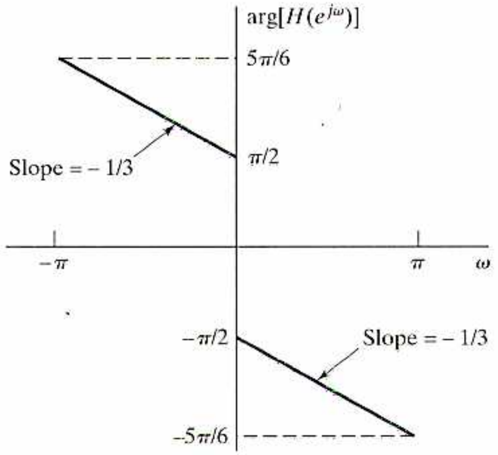

Figure P2.33-1

2.34. The input–output pair shown in Figure P2.34-1 is given for a stable LTI system.

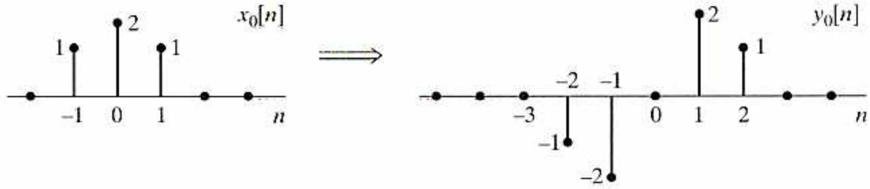

Figure P2.34-1

(a) Determine the response to the input $x_1[n]$ in Figure P2.34-2.

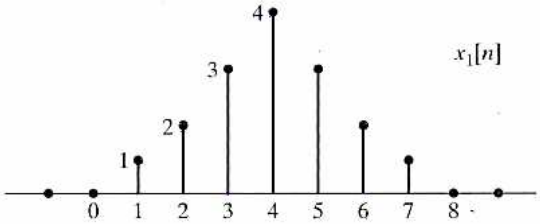

Figure P2.34-2

(b) Determine the impulse response of the system.

Advanced Problems

2.35. The system T in Figure P2.35-1 is known to be *time invariant*. When the inputs to the system are $x_1[n]$, $x_2[n]$, and $x_3[n]$, the responses of the system are $y_1[n]$, $y_2[n]$, and $y_3[n]$, as shown.

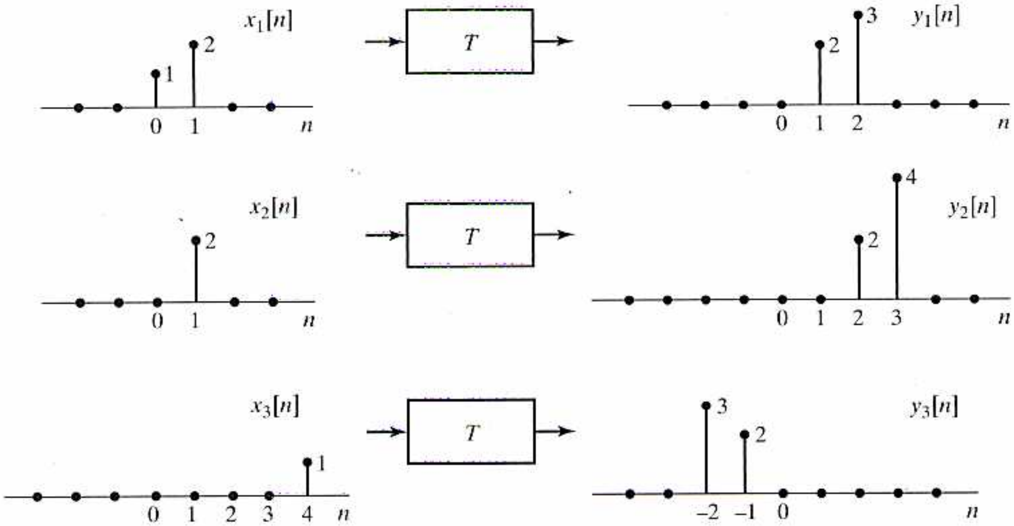

Figure P2.35-1

(a) Determine whether the system T could be linear.

(b) If the input $x[n]$ to the system T is $\delta[n]$, what is the system response $y[n]$?

(c) What are all possible inputs $x[n]$ for which the response of the system T can be determined from the given information alone?

2.36. The system L in Figure P2.36-1 is known to be *linear*. Shown are three output signals $y_1[n]$, $y_2[n]$, and $y_3[n]$ in response to the input signals $x_1[n]$, $x_2[n]$, and $x_3[n]$, respectively.

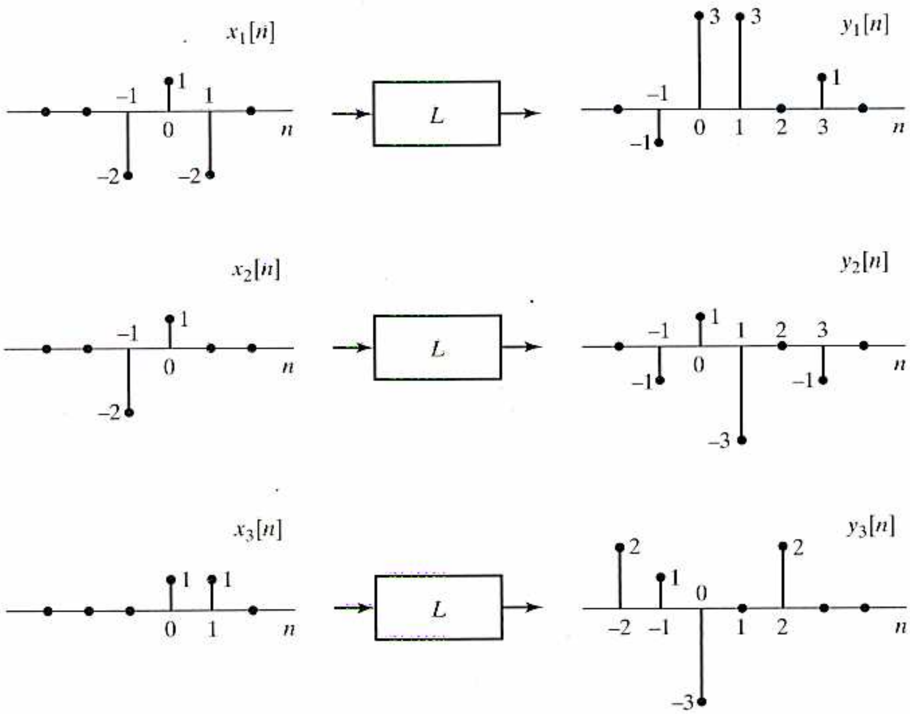

Figure P2.36-1

(a) Determine whether the system L could be time invariant.
(b) If the input $x[n]$ to the system L is $\delta[n]$, what is the system response $y[n]$?

2.37. Consider a discrete-time linear time-invariant system with impulse response $h[n]$. If the input $x[n]$ is a periodic sequence with period N (i.e., if $x[n] = x[n+N]$), show that the output $y[n]$ is also a periodic sequence with period N.

2.38. In Section 2.5, we stated that the solution to the homogeneous difference equation

$$\sum_{k=0}^{N} a_k y_h[n-k] = 0 \qquad \text{(P2.38-1)}$$

is of the form

$$y_h[n] = \sum_{m=1}^{N} A_m z_m^n, \qquad \text{(P2.38-2)}$$

with the A_m's arbitrary and the z_m's the N roots of the polynomial

$$\sum_{k=0}^{N} a_k z^{-k} = 0; \qquad \text{(P2.38-3)}$$

i.e.,

$$\sum_{k=0}^{N} a_k z^{-k} = \prod_{m=1}^{N} (1 - z_m z^{-1}). \qquad \text{(P2.38-4)}$$

(a) Determine the general form of the homogeneous solution to the difference equation

$$y[n] - \tfrac{3}{4}y[n-1] + \tfrac{1}{8}y[n-2] = 2x[n-1]. \qquad \text{(P2.38-5)}$$

(b) Determine the coefficients A_m in the homogeneous solution if $y[-1] = 1$ and $y[0] = 0$.
(c) Now consider the difference equation

$$y[n] - y[n-1] + \tfrac{1}{4}y[n-2] = 2y[n-1]. \qquad \text{(P2.38-6)}$$

If the homogeneous solution contains only terms of the form of Eq. (P2.38-2), show that the initial conditions $y[-1] = 1$ and $y[0] = 0$ cannot be satisfied.

(d) If Eq. (P2.38-3) has two roots that are identical, then, in place of Eq. (P2.38-2), $y_h[n]$ will take the form

$$y_h[n] = \sum_{m=1}^{N-1} A_m z_m^n + nB_1 z_1^n, \qquad \text{(P2.38-7)}$$

where we have assumed that the double root is z_1. Using Eq. (P2.38-7), determine the general form of $y_h[n]$ for Eq. (P2.38-6). Verify explicitly that your answer satisfies Eq. (P2.38-6) with $x[n] = 0$.

(e) Determine the coefficients A_1 and B_1 in the homogeneous solution obtained in Part (d) if $y[-1] = 1$ and $y[0] = 0$.

2.39. Consider a system with input $x[n]$ and output $y[n]$. The input–output relation for the system is defined by the following two properties:

1. $y[n] - ay[n-1] = x[n]$,
2. $y[0] = 1$.

(a) Determine whether the system is time invariant.
(b) Determine whether the system is linear.