PSYCH-GA.2211/NEURL-GA.2201 – Fall 2014
Mathematical Tools for Cognitive and Neural Science

Homework 1

Due: 19 Sep 2014
(late homeworks penalized 10% per day)

Save the solutions to each numbered problem as sections of a file called `hw1.m` in a folder called `hw1_Lastname`, along with additional files containing any functions you create. Send a zipped copy of this folder as an attachment in an email message with these attributes:

```
To: catherio@nyu.edu, asr443@nyu.edu
Subject: Math Tools HW1
```

See the course web site for full submission details. Don't wait until the day before the due date... start *now*!

1. **Inner product with a unit vector.** Given unit vector $\hat{u}$, and an arbitrary vector $\vec{x}$, write (MATLAB) expressions for computing: (a) the distance from $\vec{x}$ to the N-1 dimensional subspace (a "hyperplane") perpendicular to $\hat{u}$, (b) the component of $\vec{x}$ lying along the direction $\hat{u}$, (c) the component of $\vec{x}$ that lies in the hyperplane perpendicular to $\hat{u}$.

   Convince yourself your code is working by testing it on random vectors $\hat{u}$ and $\vec{x}$ (hint: generate these using `randn`, and don't forget to re-scale $\hat{u}$ so that it has unit length). First, do this visually with 2-dimensional vectors, by plotting $\hat{u}$, $\vec{x}$, and the two components described in (b) and (c). Then test it numerically in higher dimensions (e.g., 4) by writing (and running) expressions to verify each of the following:

   - the vector in (b) is in the same direction as $\hat{u}$
   - the length of the vector in (b) is equal to the value computed in (a)
   - the vector in (b) is perpendicular to the vector in (c)
   - the sum of the vectors in (b) and (c) is equal to $\vec{x}$.

2. **A simple linear system.** Suppose you have a retinal neuron whose response is a weighted sum of the intensities of light that land on 6 photoreceptors (note that these intensities are positive values). The weight vector is $[1, 3, 5, 4, 2, 1]$. (a) What unit-length stimulus vector elicits the largest response in the neuron? Explain how you arrived at your answer. (b) Now generate a unit-length stimulus vector that elicits a zero response in the neuron (and verify that this is true). Is this a physically realizable stimulus? Is there *any* realizable stimulus (not necessarily unit length) that would elicit a zero response in the neuron? If so, give an example.

3. **Testing for (non)linearity.** Suppose, for each of the systems below, you observe the indicated input/output pairs of vectors (or scalars). Determine whether each system could possibly be a *linear* system. If so, provide an example of a system matrix that is consistent with the observed input/output pairs. If not, explain why.

System 1:	1	$\longrightarrow$	[4, 6]
	2.5	$\longrightarrow$	[10, 15]
System 2:	[1, 2]	$\longrightarrow$	[5, -1]
	[1, -1]	$\longrightarrow$	[1, 4]
	[3, 0]	$\longrightarrow$	[7, 8]
System 3:	[6, 3]	$\longrightarrow$	[12, 12]
	[-2, -1]	$\longrightarrow$	[-6, -6]
System 4:	0	$\longrightarrow$	[1, 2]
System 5:	[2, 4]	$\longrightarrow$	0
	[-2, 1]	$\longrightarrow$	3

4. **Geometry of linear transformations**

(a) Write a function `plotVec2` that takes as an argument a matrix of height 2, and plots each column vector from this matrix on 2-dimensional axes. It should check that the matrix argument has height two, signaling an error if not. Vectors should be plotted as a line from the origin to the vector position, using circle or other symbol to denote the "head" (see help for function 'plot'). It should also draw the x and y axes, extending from -1 to 1. The two axes should be equal size, so that horizontal units are equal to vertical units (read the help for the function 'axis').

(b) Write a second function `vecLenAngle` that takes two vectors as arguments and returns the length of each vector, as well as the angle between them.

(c) Generate a random 2x2 matrix, and decompose it using the SVD. Now examine the action of this sequence of transformations (USV^T) on the two "standard basis" vectors, $\{\hat{e}_1, \hat{e}_2\}$. Specifically, use `vecLenAngle` to examine the lengths and angle between two basis vectors $\hat{e}_n$, the two vectors $V^T\hat{e}_n$, the two vectors $SV^T\hat{e}_n$, and the two vectors $USV^T\hat{e}_n$. Do these values change, and if so, after which transformation? Verify this is consistent with their visual appearance by plotting each pair using `plotVec2`.

(d) Generate a matrix P with 65 columns containing 2-dimensional unit-vectors $\hat{u}_n = [cos(\theta_n); sin(\theta_n)]$, and $\theta_n = 2\pi n/64, n = 0, 1, \ldots, 64$. [Note: Don't use a `for` loop! Create a vector containing the values of θ_n.] Plot a single blue curve through these points, and a red star (asterisk) at the location of the first point. As in the previous problem, apply the SVD transformations one at a time to full set of points (again, don't use a for loop!), plot them, and describe what geometric changes you see (and why).