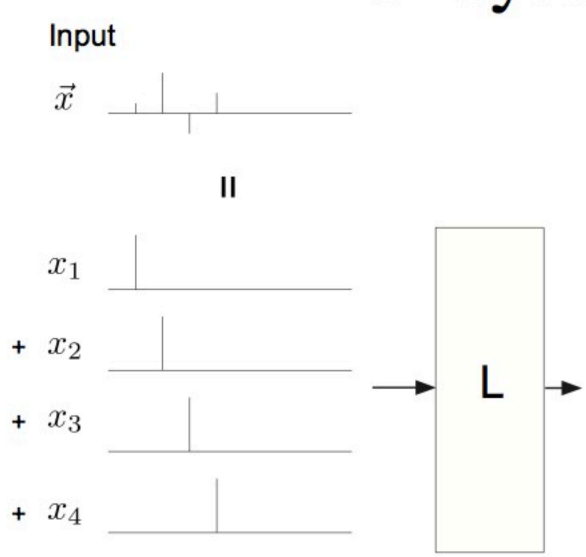
Linear shift-invariant (LSI) systems

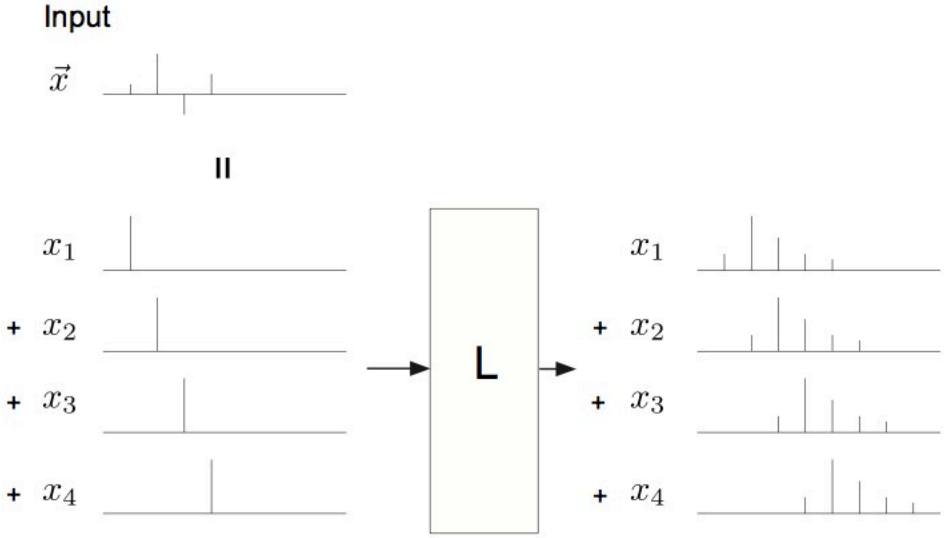
- Linearity (covered previously):
  "linear combination in, linear combination out"
- Shift-invariance (new):
  "shifted vector in, shifted vector out"
- Some examples [on board]

LSI system

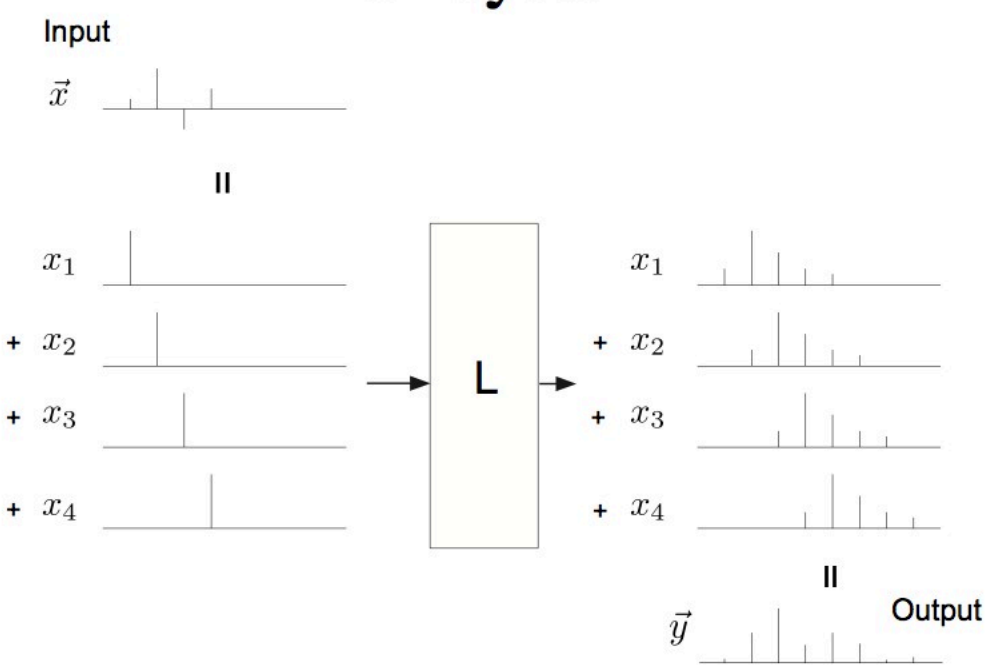


(rewrite as weighted sum of impulses)

LSI system

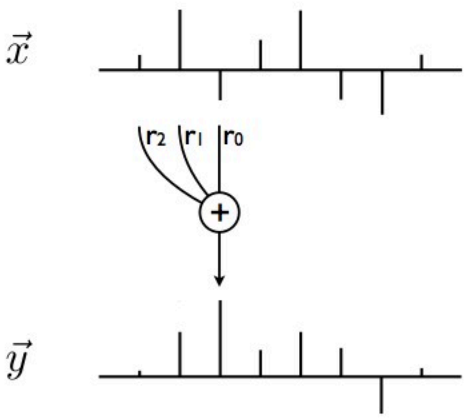


LSI system



LSI systems are characterized by their “impulse response”

Convolution

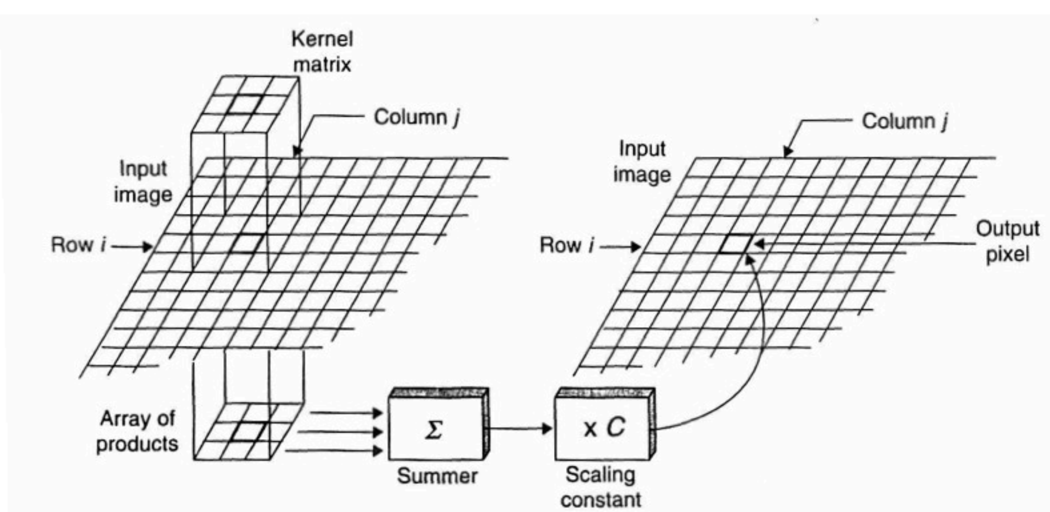


$$
\begin{aligned}
y(n) &= \sum_m r(n-m)x(m) \\
&= \sum_m r(m)x(n-m)
\end{aligned}
$$

- Matrix description
- boundaries: zero-padded, reflected, circular
- Examples: impulse, delay, average, difference

2D convolution

- sliding window



[figure c/o Castleman]

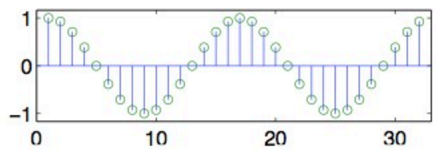
Discrete Sinusoids

$\cos(\omega n), \qquad \omega = 2\pi k/N$

"frequency" (cycles/vectorLength)

"frequency" (radians/sample)

example : $k = 2$

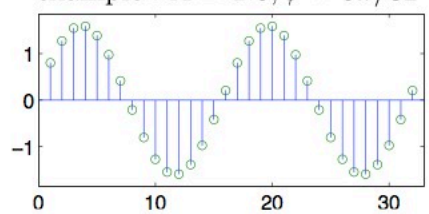


More generally: $A\cos(\omega n - \phi)$

"amplitude"

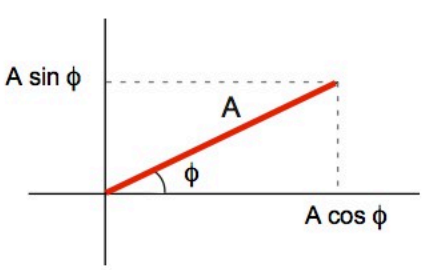
"phase" (radians)

example : $A = 1.6, \phi = 6\pi/32$

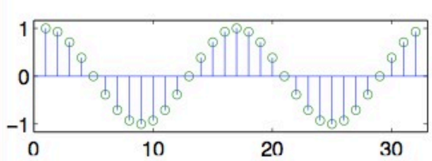


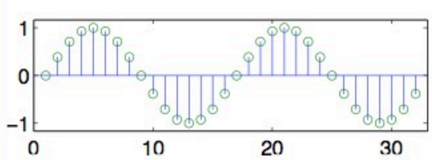
Shifting Sinusoids

$$A\cos(\omega n - \phi) = \underline{A\cos(\phi)}\,\underline{\cos(\omega n)} + \underline{A\sin(\phi)}\,\underline{\sin(\omega n)}$$



fixed cos/sin vectors:





A *shifted* sinusoidal vector can be written as a weighted sum of two *fixed* sinusoidal vectors!

Shifting Sinusoids

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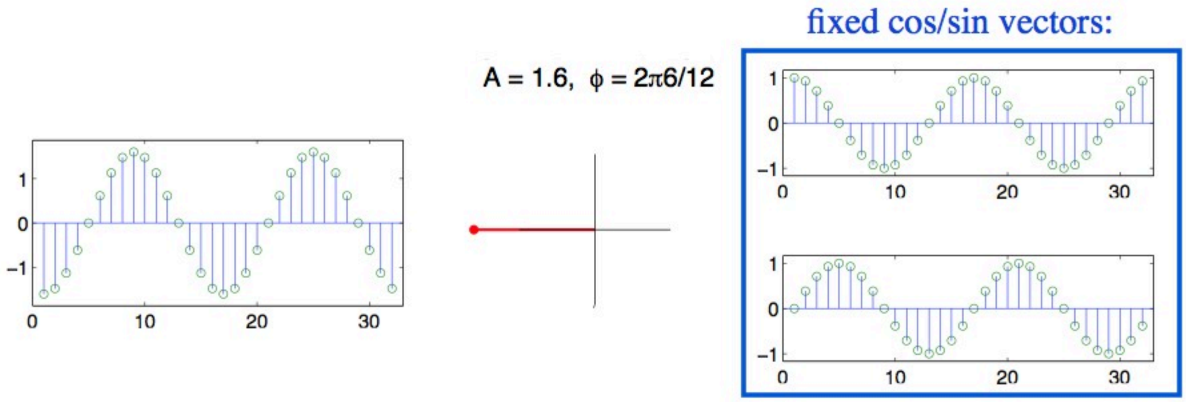

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Sinusoids & LSI

$$x(n) = \cos(\omega n)$$

$$y(n) \quad = \quad \sum_m r(m)\cos\left(\omega(n-m)\right) \qquad \text{(convolution)}$$

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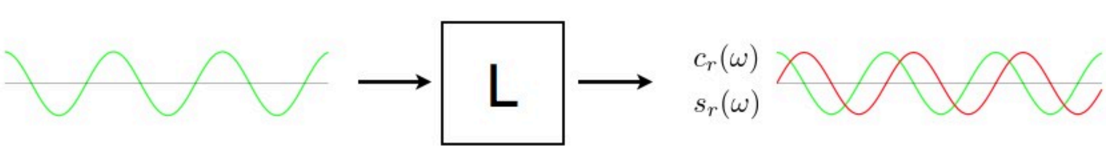
$$= c_r(\omega) \cos(\omega n) + s_r(\omega) \sin(\omega n)$$



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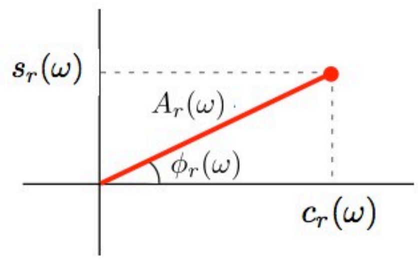
&= c_r(\omega)\cos(\omega n) + s_r(\omega)\sin(\omega n) \\
&= A_r(\omega)\cos(\phi_r(\omega))\cos(\omega n) + A_r(\omega)\sin(\phi_r(\omega))\sin(\omega n)
\end{aligned}$$



Sinusoids & LSI

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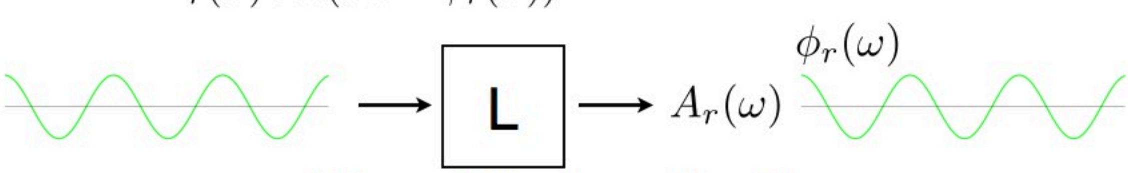
&= A_r(\omega)\cos(\phi_r(\omega))\cos(\omega n) + A_r(\omega)\sin(\phi_r(\omega))\sin(\omega n) \\
&= A_r(\omega)\cos(\omega n - \phi_r(\omega))
\end{aligned}$$



"Sinusoid in, sinusoid out"

Discrete Fourier transform (DFT)

- Construct an orthogonal matrix of sin/cos pairs, at frequency multiples of $2\pi/N$ radians/sample, i.e., $2\pi k/N, \quad \text{for } k = 0, 1, 2, \ldots N/2$
- For $k = 0$ and $k = N/2$, only need the cosine part of the pair
- When we transform a vector using this matrix, think of output as paired coordinates
- Common to plot these pairs as amplitude/phase

[all details on board]

The Fourier family

signal domain

frequency domain	continuous	discrete
continuous	**Fourier transform**	**discrete-time Fourier transform**
discrete	**Fourier series**	**discrete Fourier transform**

you are here

The "fast Fourier transform" (FFT) is a computationally efficient implementation of the DFT (runs in NlogN time, instead of N^2).

Sinusoids & LSI

$x(n) = \cos(\omega n)$

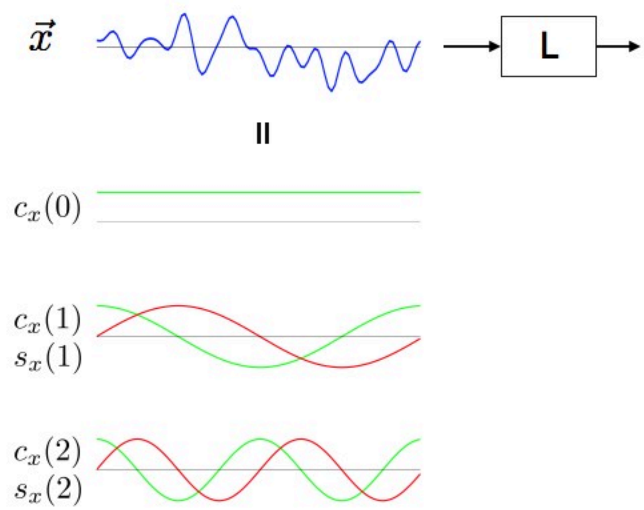
$$\begin{aligned} y(n) &= \sum_m r(m) \cos\left(\omega(n-m)\right) \\ &= \sum_m r(m)\cos(\omega m) \cos(\omega n) + \sum_m r(m)\sin(\omega m) \sin(\omega n) \\ &= c_r(\omega) \cos(\omega n) + s_r(\omega) \sin(\omega n) \\ &= A_r(\omega)\cos(\phi_r(\omega))\cos(\omega n) + A_r(\omega)\sin(\phi_r(\omega))\sin(\omega n) \\ &= A_r(\omega)\cos(\omega n - \phi_r(\omega)) \end{aligned}$$

NOTE: Change in amplitude and phase come from Fourier transform of the impulse response!

Fourier & LSI

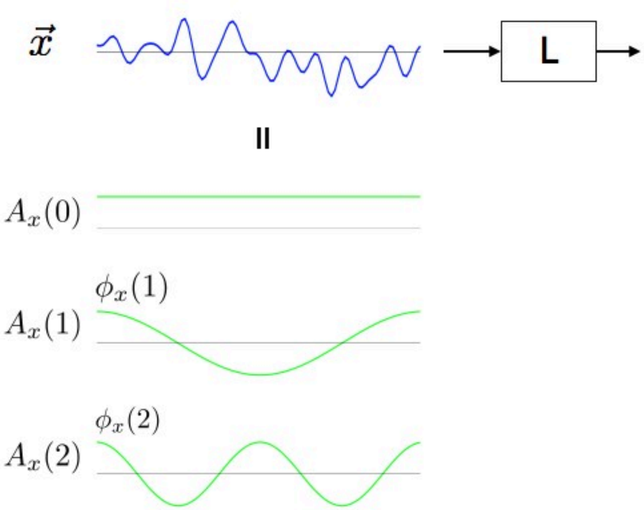
$\vec{x}$ 

Fourier & LSI



note: only 3 (of many) components shown

Fourier & LSI



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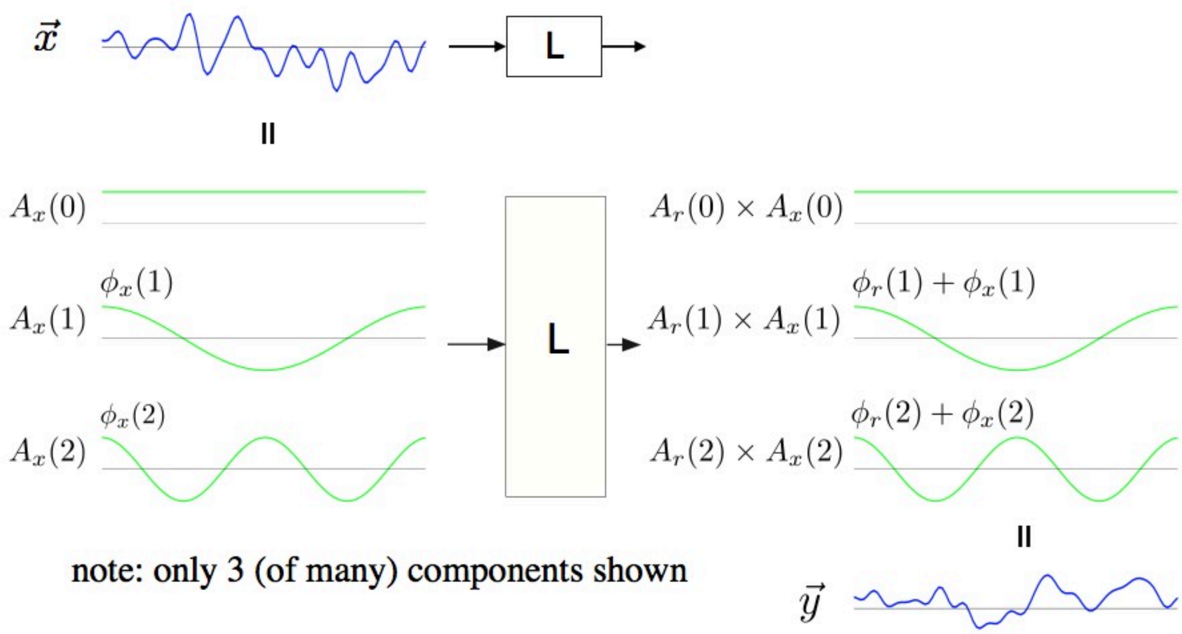
Fourier & LSI



note: only 3 (of many) components shown

LSI systems are characterized by their *frequency response*, specified by the Fourier Transform of their impulse response

Complex exponentials: "bundling" sine and cosine

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$e^{i\omega n} \longrightarrow \boxed{L} \longrightarrow A_r(\omega)\ e^{i(\omega n - \phi_r(\omega))} \quad = \quad A_r(\omega)\ e^{-i\phi_r(\omega)}\ e^{i\omega n}$$

$$= \quad \tilde{r}(\omega)\ e^{i\omega n}$$

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F.T. of impulse response!

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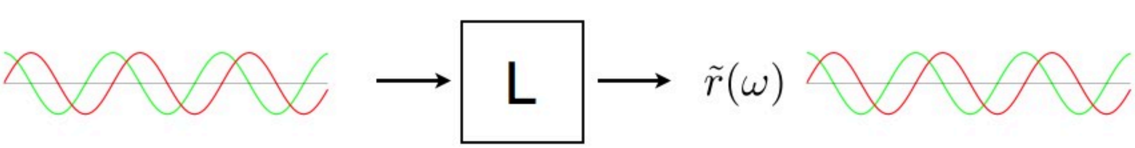
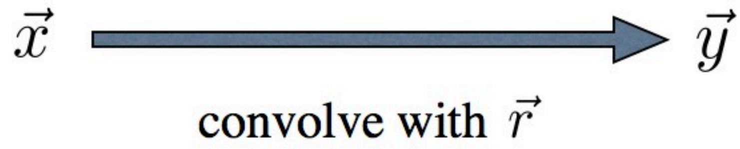


The "convolution theorem"



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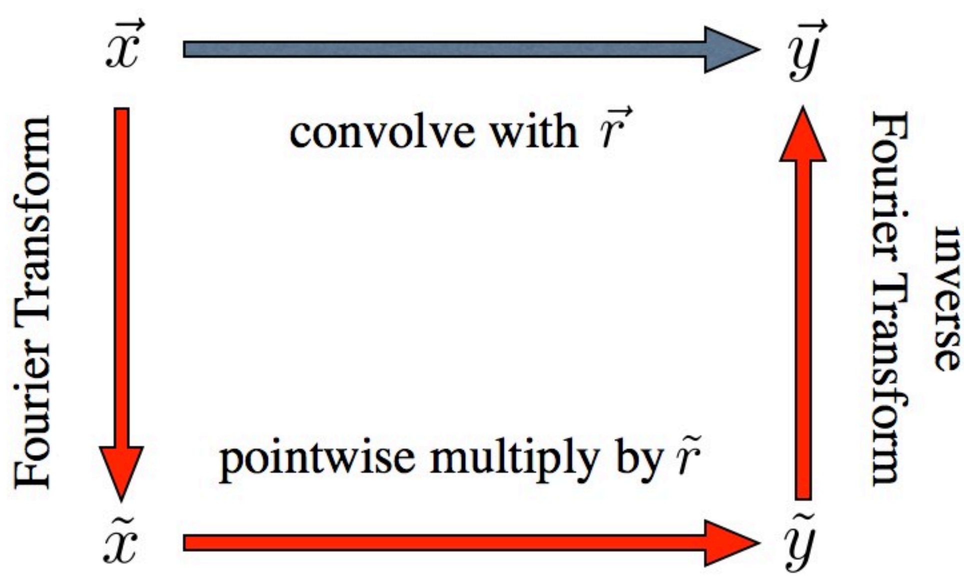

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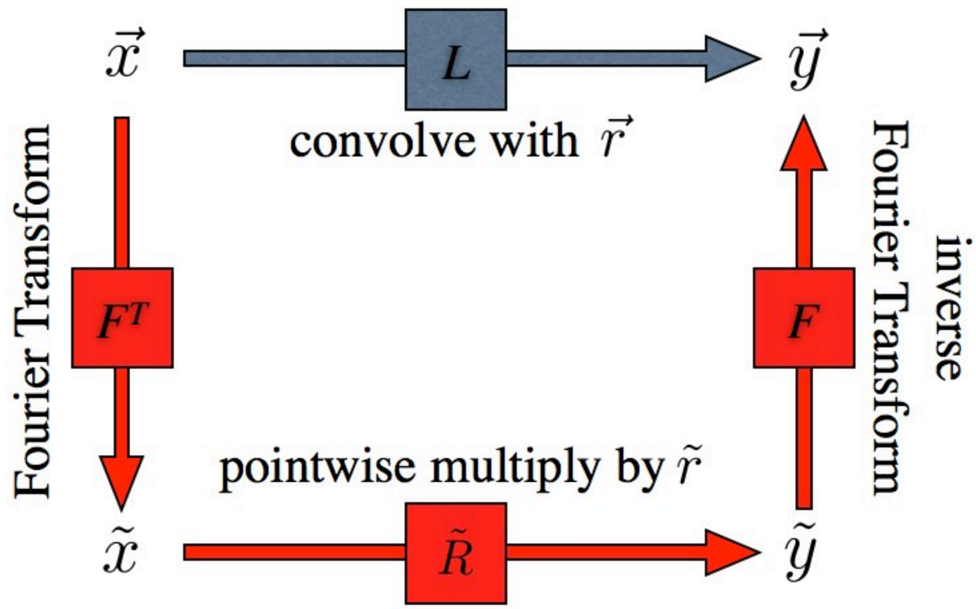


In matrix form: $L\vec{x} = F\tilde{R}F^T\vec{x}$

Recap

- Linear system

  => defined by superposition

  => characterized by a matrix

- Linear Shift-invariant (LSI) system

  => defined by superposition and shift-invariance

  => characterized by impulse response

  => alternatively, characterized by frequency response (the Fourier Transform of the impulse response!)

Discrete Fourier transform (with complex numbers)

$$\tilde{r}_k = \sum_{n=0}^{N-1} r_n e^{-i\omega_k n}$$

$$r_n = \sum_{k=0}^{N-1} \tilde{r}_k \; e^{i\omega_k n} \qquad \text{(inverse)}$$

where $\omega_k = \dfrac{2\pi k}{N}$

Visualizing the (discrete) Fourier transform

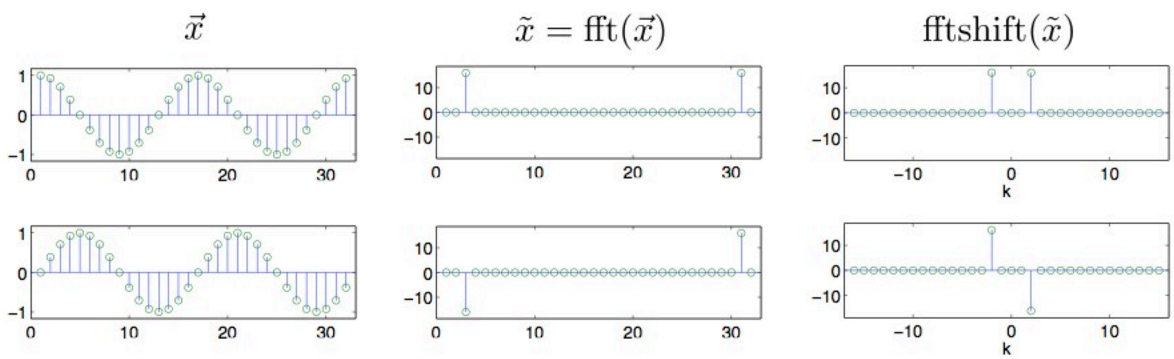
- Typically, plot Amplitude (and possibly Phase, on a separate graph), instead of real/imaginary parts
- Two conventional choices for frequency axis:
  - Plot frequencies from k=0 to k=N/2
  - Plot frequencies from k=-N/2 to N/2-1

Dealing with DFT in Matlab...

$$e^{i\omega n} = \cos(\omega n) + i\sin(\omega n)$$

$$\cos(\omega n) = \frac{1}{2}(e^{i\omega n} + e^{-i\omega n}) \qquad \sin(\omega n) = \frac{-i}{2}(e^{i\omega n} - e^{-i\omega n})$$

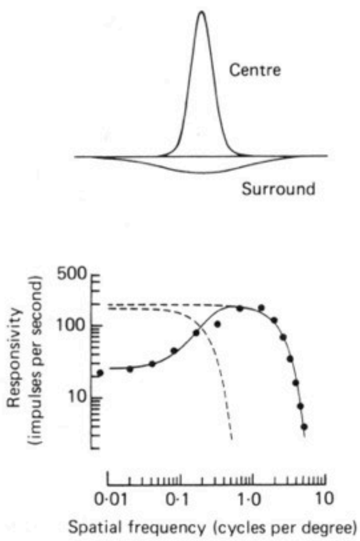
Matlab examples, N=32 (note indexing and amplitudes):

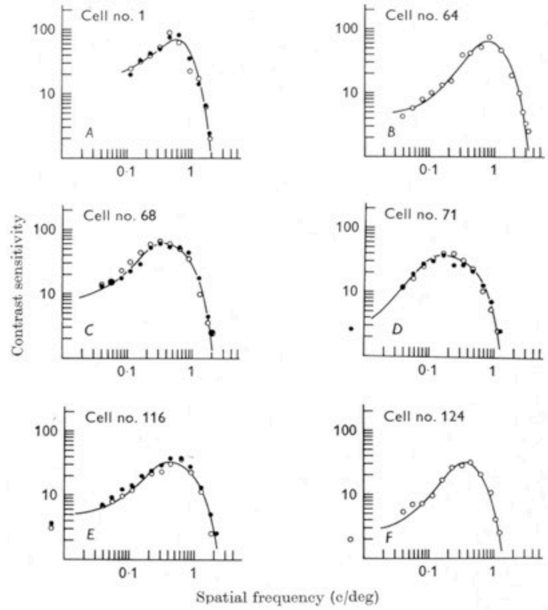


Some examples

- constant
- sinusoid
- impulse
- Gaussian - “lowpass”
- DoG (difference of 2 Gaussians) - “bandpass”
- Gabor (Gaussian windowed sinusoid) - “bandpass”

Retinal ganglion cells (1D)





Enroth-Cugell and Robson (1984)