Mathematical Tools
for Neural and Cognitive Science

Fall semester, 2025

Section 4:
Summary Statistics & Probability

Statistics is the science of learning from experience, especially experience that arrives a little bit at a time. The earliest information science was statistics, originating in about 1650. This century has seen statistical techniques become the analytic methods of choice in biomedical science, psychology, education, economics, communications theory, sociology, genetic studies, epidemiology, and other areas. Recently, traditional sciences like geology, physics, and astronomy have begun to make increasing use of statistical methods as they focus on areas that demand informational efficiency, such as the study of rare and exotic particles or extremely distant galaxies.

Most people are not natural-born statisticians. Left to our own devices we are not very good at picking out patterns from a sea of noisy data. To put it another way, we are all too good at picking out non-existent patterns that happen to suit our purposes. Statistical theory attacks the problem from both ends. It provides optimal methods for finding a real signal in a noisy background, and also provides strict checks against the overinterpretation of random patterns.

[Efron & Tibshirani, 1998]

Some historical context

- 1600's: Early notions of data summary/averaging
- 1700's: Bayesian prob/statistics (Bayes, Laplace)
- 1920's: Frequentist statistics for science (e.g., Fisher)
- 1940's: Statistical signal analysis and communication, estimation/decision theory (e.g., Shannon, Wiener, etc)
- 1950's: Return of Bayesian statistics (e.g., Jeffreys, Wald, Savage, Jaynes…)
- 1970's: Computation, optimization, simulation (e.g,. Tukey)
- 2000's: Machine learning (statistical inference with large-scale computing + lots of data)
- Also (since 1950's): statistical neural/cognitive models!

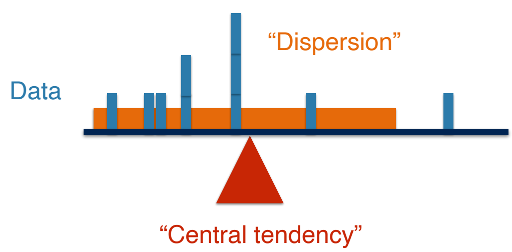
Statistics as summary description

0.1, 4.5, -2.3, 0.8, -1.1, 3.2, …

"The purpose of statistics is to replace a quantity of data by relatively few quantities which shall ... contain as much as possible, ideally the whole, of the relevant information contained in the original data"

- R.A. Fisher, 1934

Standard descriptive statistics

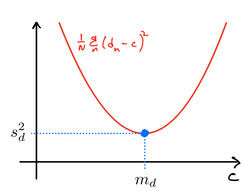


Descriptive statistics: 1D

The most common measures of central tendency & dispersion:

- Sample mean ***minimizes*** the squared error

$$m_d = \arg\min_c \frac{1}{N} \sum_{n=1}^{N} (d_n - c)^2$$
$$= \frac{1}{N} \sum_n d_n$$



- Sample variance ***is*** the squared error

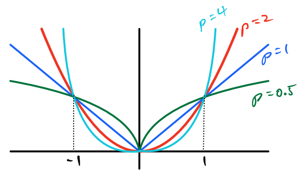
$$s_d^2 = \min_c \frac{1}{N} \sum_{n=1}^{N} (d_n - c)^2 = \frac{1}{N} \sum_n (d_n - m_d)^2$$
$$= \frac{1}{N} \sum_n d_n^2 - m_d^2$$

(second moment minus squared mean)

Descriptive statistics: generalizations

More generally, can measure **dispersion** with an "L_p norm":

$$\left[\sum_{n=1}^{N} |d_n - c|^p\right]^{1/p}$$

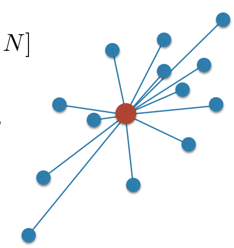


Different p values give different measures of **central tendency**:

- $p = 2$: mean (standard choice)
- $p = 1$: median
- $p \to 0$: mode (location of maximum)
- $p \to \infty$: midpoint of range

Descriptive statistics: 2-D

- Data points: $\vec{d}_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}, \quad n \in [1 \ldots N]$
- Sample mean: the vector that minimizes average squared distance to data points:



$$\vec{m}_d = \arg\min_{\vec{c}} \frac{1}{N} \sum_{n=1}^{N} \|\vec{d}_n - \vec{c}\|^2, \quad \vec{c} = \begin{bmatrix} c_x \\ c_y \end{bmatrix}$$

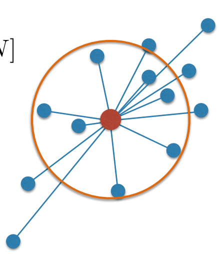
$$= \arg\min_{c_x, c_y} \frac{1}{N} \sum_n \left[(x_n - c_x)^2 + (y_n - c_y)^2\right]$$

$$= \frac{1}{N} \sum_n \begin{bmatrix} x_n \\ y_n \end{bmatrix} = \frac{1}{N} \sum_n \vec{d}_n$$

(analogous to scalar case!)

Descriptive statistics: 2-D

- Data points: $\vec{d}_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}, \quad n \in [1 \ldots N]$
- Sample mean:



$$\vec{m}_d = \arg\min_{\vec{c}} \frac{1}{N} \sum_{n=1}^{N} \|\vec{d}_n - \vec{c}\|^2 = \frac{1}{N} \sum_n \vec{d}_n$$

- Sample (total) variance:

$$s_d^2 = \min_{\vec{c}} \frac{1}{N} \sum_{n=1}^{N} \|\vec{d}_n - \vec{c}\|^2 = \frac{1}{N} \sum_n \|\vec{d}_n - \vec{m}_d\|^2$$

$$= \frac{1}{N} \sum_n \|\vec{d}_n\|^2 - \|\vec{m}_d\|^2$$

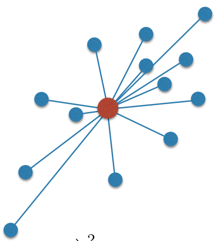
(analogous to scalar case!)

Descriptive statistics: 2-D

- Sample mean, in direction $\hat{u}$

$$m_u = \arg\min_c \frac{1}{N}\sum_n \left(\hat{u}^T \vec{d}_n - c\right)^2$$

$$= \frac{1}{N}\sum_n \hat{u}^T \vec{d}_n = \hat{u}^T \frac{1}{N}\sum_n \vec{d}_n = \boxed{\hat{u}^T \vec{m}_d}$$



- Sample variance, in direction $\hat{u}$

$$s_u^2 = \min_c \frac{1}{N}\sum_{n=1}^{N} \left(\hat{u}^T \vec{d}_n - c\right)^2 = \frac{1}{N}\sum_n \left(\hat{u}^T \vec{d}_n - \hat{u}^T \vec{m}_d\right)^2$$

$$= \hat{u}^T \boxed{\left[\frac{1}{N}\sum_n \left(\vec{d}_n - \vec{m}_d\right)\left(\vec{d}_n - \vec{m}_d\right)^T\right]} \hat{u}$$

sample covariance, C_d

$$= \hat{u}^T \left[\frac{1}{N}\sum_n \left(\vec{d}_n \vec{d}_n^T - \vec{m}_d \vec{m}_d^T\right)\right] \hat{u}$$

Descriptive statistics: 2-D

- Sample mean, in direction $\hat{u}$

$$m_u = \arg\min_c \frac{1}{N}\sum_n \left(\hat{u}^T \vec{d}_n - c\right)^2$$

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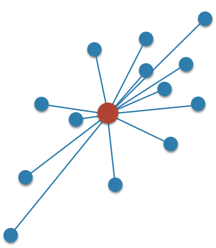


- Sample variance, in direction $\hat{u}$

$$s_u^2 = \hat{u}^T \boxed{\left[\frac{1}{N}\sum_n \left(\vec{d}_n \vec{d}_n^T - \vec{m}_d \vec{m}_d^T\right)\right]} \hat{u}$$

sample covariance, C_d

$$\boxed{\frac{1}{N}\sum_n \begin{bmatrix} x_n^2 & x_n y_n \\ y_n x_n & x_n^2 \end{bmatrix} - \begin{bmatrix} m_x^2 & m_x m_y \\ m_y m_x & m_y^2 \end{bmatrix}} = \begin{bmatrix} s_x^2 & s_{xy} \\ s_{xy} & s_y^2 \end{bmatrix}$$

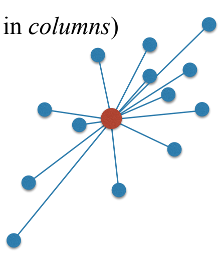
Descriptive statistics: multi-D

- Data points: matrix D (N data vectors in *columns*)

- Sample mean:

$$\vec{m}_d = \frac{1}{N}\sum_n \vec{d}_n = \frac{1}{N} D \vec{1}$$

vector of N ones



- Sample variance, in direction $\hat{u}$

$$s_u^2 = \frac{1}{N}\sum_n \left((\vec{d}_n - \vec{m}_d)^T \hat{u}\right)^2 = \frac{1}{N}\|\boxed{(D - \vec{m}_d \vec{1}^T)}^T \hat{u}\|^2$$

D^*

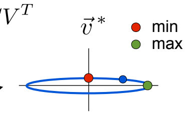
$$= \frac{1}{N}\|D^{*T}\hat{u}\|^2$$

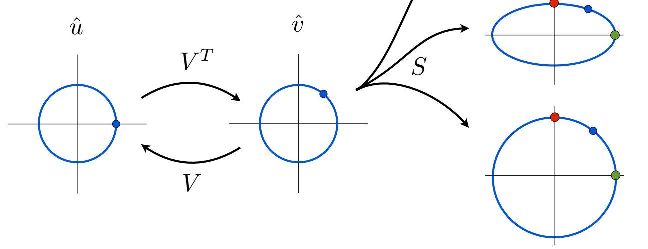
As we vary direction $\hat{u}$, what does the sample variance do?

Recall: rewrite $\|D^{*T}\hat{u}\|^2$ with SVD $D^{*T} = USV^T$

$$\|USV^T\hat{u}\|^2 = \|SV^T\hat{u}\| = \|S\hat{v}\|^2 = \|\vec{v}^*\|^2,$$

$$\text{where } \hat{v} = V^T\hat{u}, \quad \text{and } \vec{v}^* = S\hat{v}$$





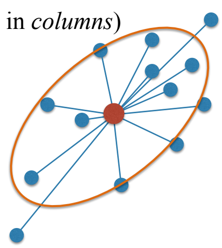
Set of $\hat{u}$'s of length 1 (i.e., unit vectors)

Corresponding set of $\hat{v}$'s

Corresponding set of $\vec{v}^*$'s

Descriptive statistics: multi-D

- Data points: matrix D (N data vectors in *columns*)



- Sample mean, in direction $\hat{u}$:

$$m_u = \frac{1}{N}(\hat{u}^T D)\vec{1} = \hat{u}^T \left[\frac{D\,\vec{1}}{N}\right]$$

sample mean, $\vec{m}_d$

- Sample variance, in direction $\hat{u}$:

$$s_u^2 = \frac{1}{N}\sum_n \left((\vec{d}_n - \vec{m}_d)^T\hat{u}\right)^2 = \frac{1}{N}\|(D - \vec{m}_d\vec{1}^T)^T\hat{u}\|^2$$

$$= \frac{1}{N}\|D^{*T}\hat{u}\|^2 = \hat{u}^T\left[\frac{D^*D^{*T}}{N}\right]\hat{u}$$

sample covariance, C_d

Principal Component Analysis (PCA)

The shape of a data cloud can be summarized with an ellipse (ellipsoid), centered around the mean, using a simple procedure:

(1) Subtract mean from all data points (re-centers data around origin)

(2) Collect centered data vectors in columns of a matrix, D^*

(3) Compute the SVD: $D^{*T} = USV^T$

or use covariance matrix $C_d = D^*D^{*T} = VS^TSV^T$

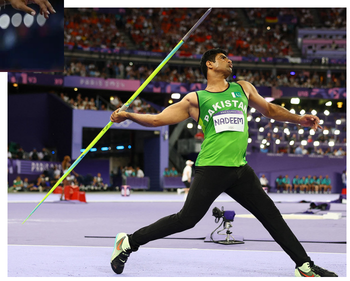
Λ, square and diagonal, elements λ_k

- Columns of V are the *principal components* (axes) of the ellipsoid, singular values s_k (or $\sqrt{\lambda_k}$) are the corresponding *principal radii.*
- Ellipse volume is proportional to product of s_k's.
- Total variance is equal to sum of λ_k's.

Olympic gold medalists
(Paris, 2024)

Valerie Allman (USA)

Yemisi Ogunleye (Germany)



Arshad Nadeem (Pakistan)

3D geometry:
shotput, discus, javelin…

Eigenvectors/eigenvalues

- An *eigenvector* of a matrix is a vector that is rescaled by the matrix (i.e., the direction is unchanged)
- The corresponding scale factor is called the *eigenvalue*

- For covariance matrix $C_d = D^* D^{*T} = V \Lambda V^T$ the columns of V (denoted $\hat{v}_k$) are eigenvectors, with corresponding eigenvalues λ_k:

$$\begin{aligned} C_d \hat{v}_k &= V \Lambda V^T \hat{v}_k \\ &= V \Lambda \hat{e}_k \\ &= \lambda_k V \hat{e}_k \\ &= \lambda_k \hat{v}_k \end{aligned}$$

- For LSI system L, the eigenvectors are complex exponentials:

$$L \vec{v}_k = F R F^T \vec{v}_k = r_k \vec{v}_k$$

F is the Fourier transform, $\vec{v}_k$ the kth Fourier basis function, r_k the kth entry of diagonal matrix R containing F.T. of impulse response

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Affine transformations

If $\vec{b}_n = M \left(\vec{d}_n - \vec{a} \right)$ (translate, then rotate-stretch-rotate)

then $\vec{m}_b = M \left(\vec{m}_d - \vec{a} \right)$ (mean and covariance transform according to simple rules)

$C_b = M C_d M^T$

Standard case: "re-center" and "normalize" the components:

Let $\vec{a} = \vec{m}_d$ $\quad M = \begin{bmatrix} \frac{1}{s_x} & 0 \\ 0 & \frac{1}{s_y} \end{bmatrix}$

then $\vec{m}_b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad C_b = \begin{bmatrix} 1 & \frac{s_{xy}}{s_x s_y} \\ \frac{s_{xy}}{s_x s_y} & 1 \end{bmatrix}$

"r" (Pearson correlation coefficient)

[on board]

Correlation (r) captures dependency

… but not slope!

Regression (revisited)

$$\vec{e} = \vec{y} - \beta\vec{x}$$

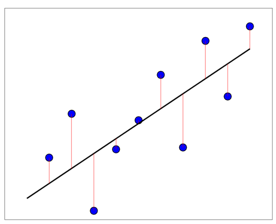
Optimal regression line slope:

$$\beta = \frac{\vec{x}^T\vec{y}}{\vec{x}^T\vec{x}} = \frac{s_{xy}}{s_x^2}$$

Error variance:

$$s_e^2 = s_y^2 - 2\beta s_{xy} + \beta^2 s_x^2$$

$$= s_y^2 - \frac{s_{xy}^2}{s_x^2}$$



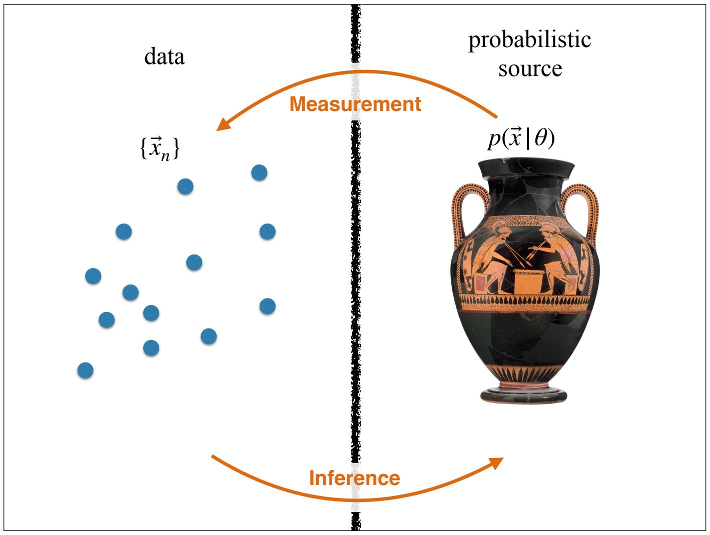
Expressed as a proportion of σ_y^2:

$$\frac{s_e^2}{s_y^2} = 1 - \frac{s_{xy}^2}{s_x^2 s_y^2} = 1 - r^2$$

proportion of data variance explained

Probability: an abstract mathematical framework for describing random quantities.

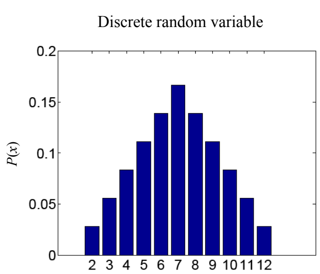
Statistics: use of probability to summarize, analyze, and interpret data. **Fundamental to all experimental science.**

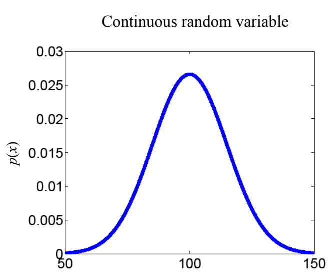


Univariate Probability (outline)

- distributions: discrete and continuous
- expected value, moments
- transformations: affine, monotonic nonlinear
- cumulative distributions. Quantiles, drawing samples

Probability distributions

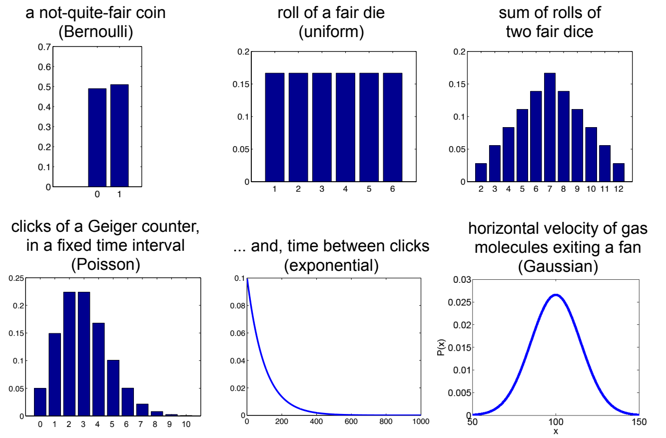


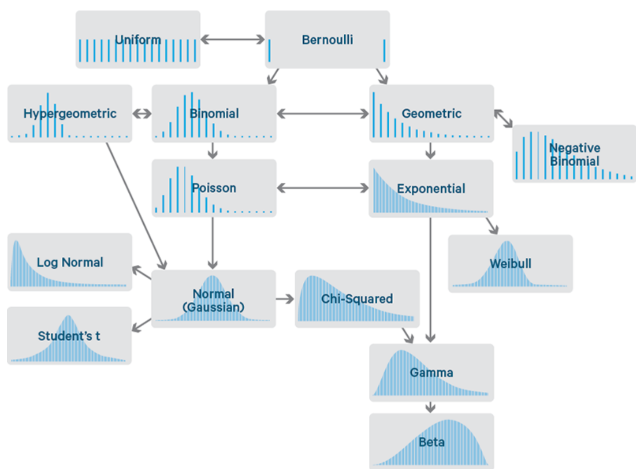


$$0 \leq P(x_i) \leq 1, \quad \forall i$$
$$\sum_i P(x_i) = 1$$

$$0 \leq p(x)$$
$$\int_{-\infty}^{\infty} p(x)\, dx = 1$$

Example distributions



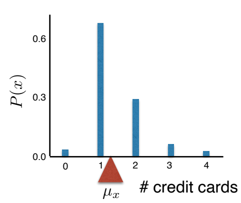


[Figure: Sean Owen, Cloudera Engineering]

Expected value (for a discrete random variable)

$$\mu_x = \mathbb{E}(x) = \sum_{k=1}^{K} x_k \; P(x_k)$$

a weighted sum over the discrete values



More generally: $$\mathbb{E}\left(f(x)\right) = \sum_{k=1}^{K} f(x_k) P(x_k)$$ (sum over values of R.V.)

Sample average: an estimate of the expected value:

$$\bar{f}(x) = \frac{1}{N} \sum_{n=1}^{N} f(x_n)$$ (sum over data samples)

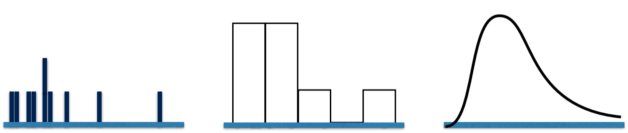
Sample average converges to expected value as one gathers more data…

A note on notation

- We have, and will continue to use the notation for a "sample mean" ($\bar{x}$) and a "sample standard deviation" (s) or variance (s^2).
- Statistics makes a distinction between these sample values and the corresponding "population" values of mean (μ) and variance (σ^2).

Connecting statistics and probability (limit of infinite data)

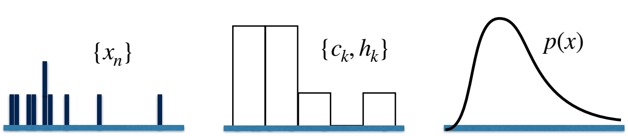
data → histogram → probability distribution



$\{x_n\}$ $\{c_k, h_k\}$ $p(x)$

Expected value (continuous random variable)

data → histogram → probability distribution



{center, normalized height} of kth bin

$$\bar{x} = \frac{1}{N}\sum_n x_n \qquad \bar{x} \approx \sum_k c_k h_k = \vec{c}^T\vec{h} \qquad \mu_x = \int x\ p(x)\ dx$$

Expected value (continuous)

$$\mathbb{E}(x) = \int x \; p(x) \; dx \qquad [\text{“mean”}, \; \mu]$$

$$\mathbb{E}(x^2) = \int x^2 p(x) \; dx \qquad [\text{“second moment”}, \; m_2]$$

$$\mathbb{E}\left((x-\mu)^2\right) = \int (x-\mu)^2 \; p(x) \; dx \qquad [\text{“variance”}, \; \sigma^2]$$

$$= \int x^2 p(x) \; dx - \mu^2 \qquad [m_2 \textit{ minus } \mu^2]$$

$$\mathbb{E}\left(f(x)\right) = \int f(x) \; p(x) \; dx \qquad [\text{“expected value of } f\text{”}]$$

Note: expectation is an integral, and thus *linear*, so:

$$\mathbb{E}\left(af(x) + bg(x)\right) = a\mathbb{E}\left(f(x)\right) + b\mathbb{E}\left(g(x)\right)$$

Transformations of scalar random variables

$Y = aX + b$ “affine” (linear plus constant)

Analogous to sample mean/covariance:

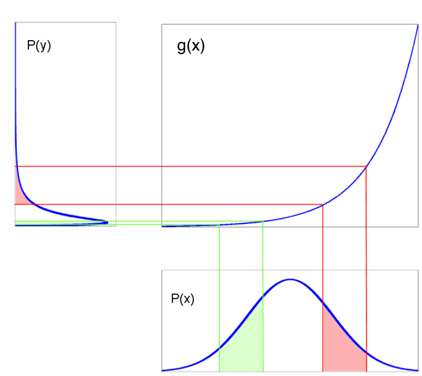
$$\mu_Y = \mathbb{E}(Y) = a\mathbb{E}(X) + b = a\mu_X + b$$

$$\sigma_Y^2 = \mathbb{E}\left(\left(Y - \mu_Y\right)^2\right) = \mathbb{E}\left(\left(aX - a\mu_X\right)^2\right) = a^2\sigma_X^2$$

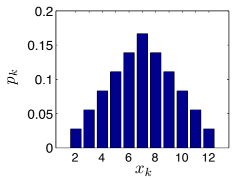
Full distribution: $p_Y(y) = \frac{1}{a} \; p_X\left(\frac{y-b}{a}\right)$

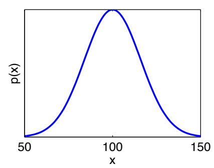
$Y = g(X)$ (assume g is “monotonic” - i.e., derivative > 0)

$$p_Y(y) = \frac{p_X\left(g^{-1}(y)\right)}{g'\left(g^{-1}(y)\right)}$$



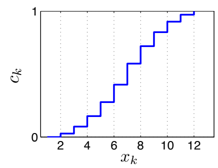
Cumulative distributions

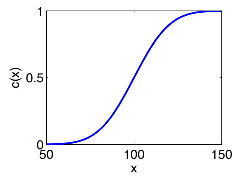




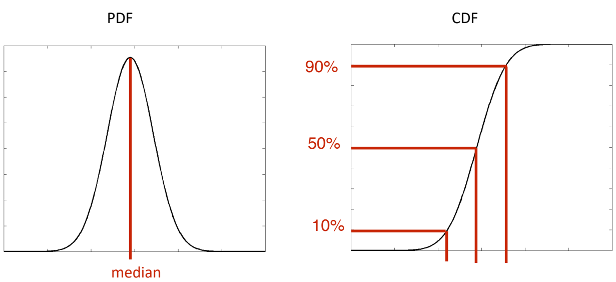
$$c_k = \sum_{j=-\infty}^{k} p_j$$

$$c(x) = \int_{-\infty}^{x} p(z)\ dz$$

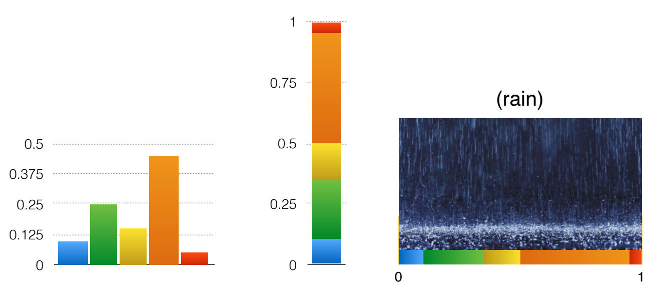




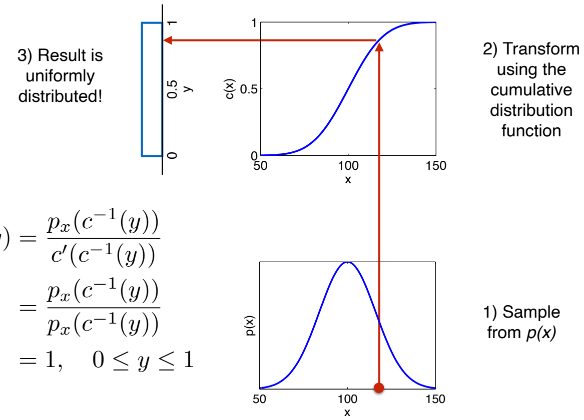
Quantiles



Drawing samples - discrete

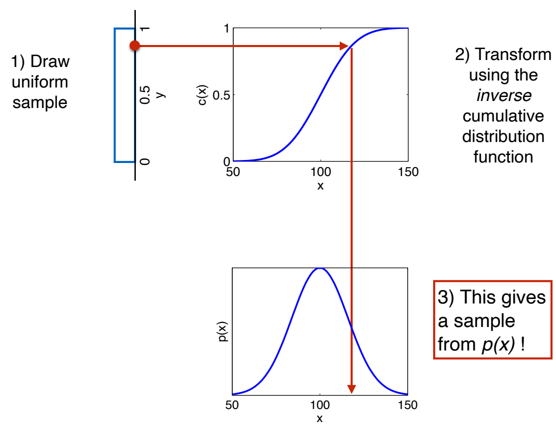


Drawing samples - continuous



$$p_y(y) = \frac{p_x(c^{-1}(y))}{c'(c^{-1}(y))} = \frac{p_x(c^{-1}(y))}{p_x(c^{-1}(y))} = 1, \quad 0 \le y \le 1$$

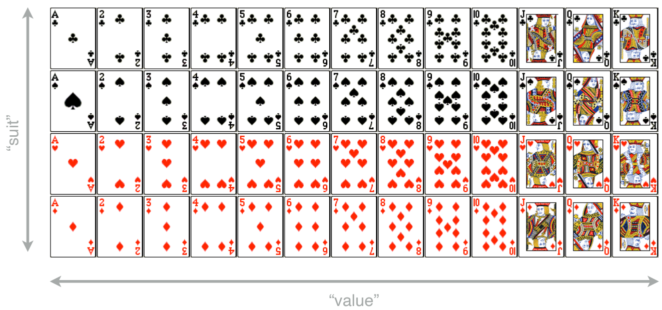
Drawing samples - continuous



Multi-variate probability (outline)

- Joint distributions
- Marginals (integrating)
- Conditionals (slicing)
- Bayes' rule (inverse probability)
- Statistical independence (separability)
- Mean/Covariance
- Linear transformations

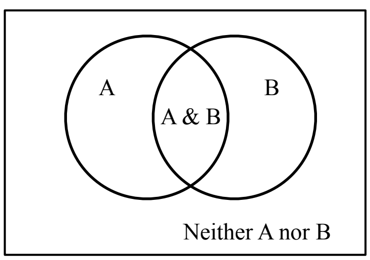
Joint probability - discrete



P(Ace)
P(Heart)

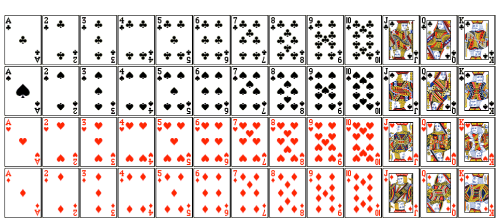
P(Ace & Heart)
P(Ace & red)
P(number & Heart)

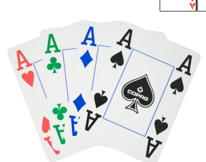
Conditional probability



$p(A\,|\,B)$ = probability of A given that B is asserted to be true = $\dfrac{p(A\,\&\,B)}{p(B)}$

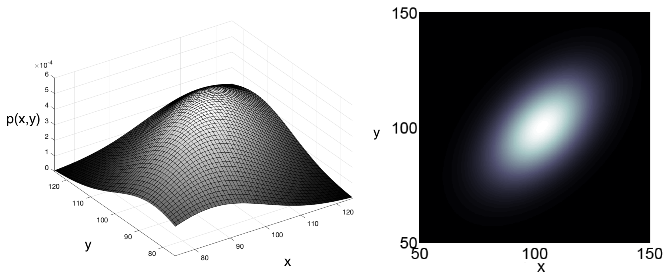
Conditional probability - discrete





P(Ace | Heart)
P(Ace | red)
P(number | Heart)

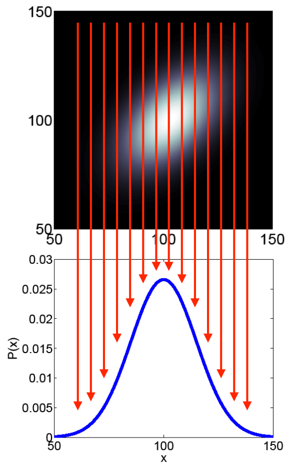
Joint distribution (continuous)



$$p(x, y)$$

Marginal distribution

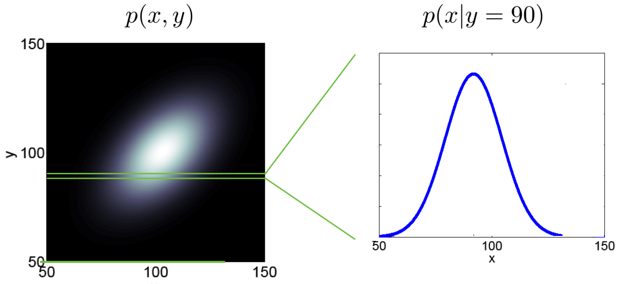
$$p(x, y)$$



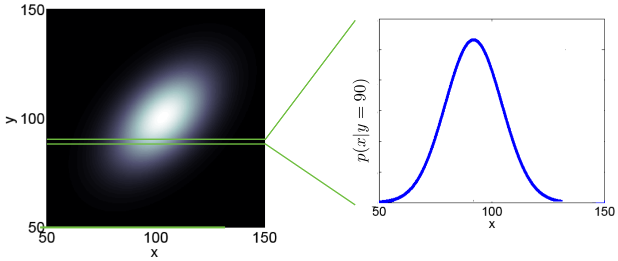
$$p(x) = \int p(x, y) dy$$

Conditional distribution

$p(x, y)$ $\qquad$ $p(x|y = 90)$



Conditional distribution



$$p(x|y=90) = p(x, y=90) \Big/ \int p(x, y=90)dx$$

$$= p(x, y=90) / p(y=90)$$

slice joint distribution — normalize (by marginal)

More generally:

$$p(x|y) = p(x,y)/p(y)$$

Bayes' Rule

LII. *An Essay towards solving a Problem in the Doctrine of Chances. By the late Rev. Mr.* Bayes, *F. R. S. communicated by Mr.* Price, *in a Letter to* John Canton, *A. M. F. R. S.*

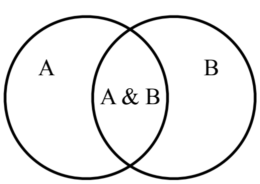
Dear Sir,

Read Dec. 23, 1763. I Now send you an essay which I have found among the papers of our deceased friend Mr. Bayes, and which, in my opinion, has great merit, and well deserves to be preserved.

$$p(x|y) = p(y|x)\,p(x)/p(y)$$

(a direct consequence of the definition of conditional probability)

Bayes' Rule



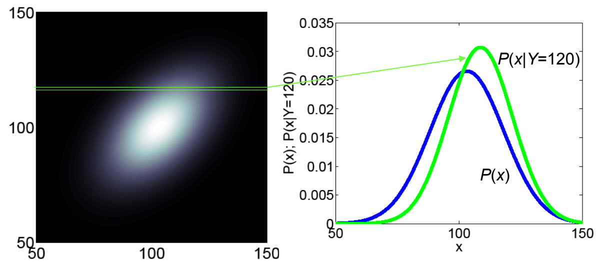
$p(A\,|\,B) =$ probability of A given that B is asserted to be true $= \dfrac{p(A \,\&\, B)}{p(B)}$

$$p(A \,\&\, B) = p(B)p(A\,|\,B)$$

$$= p(A)p(B\,|\,A)$$

$$\Rightarrow p(A\,|\,B) = \frac{p(B\,|\,A)p(A)}{p(B)}$$

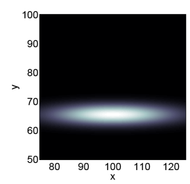
Conditional vs. marginal



In general, the marginals for different Y values differ.

When are they they same? In particular, when are all conditionals equal to the marginal?

Statistical independence



Random variables X and Y are statistically independent if (and only if):

$$p(x,y) = p(x)p(y) \quad \forall\ x,y$$

(note: for discrete distributions, this is an outer product!)

Independence implies that *all* conditionals are equal to the corresponding marginal:

$$p(x \mid y) = p(x,y)\,/\,p(y) = p(x) \quad \forall\ x,y$$

Mean, covariance, affine transformations

For R.V. $\vec{X}$, $\vec{\mu}_X = \mathbb{E}(\vec{X})$, $C_X = \mathbb{E}\left((\vec{X} - \vec{\mu}_X)(\vec{X} - \vec{\mu}_X)^T\right)$

For R.V. $\vec{Y} = M(\vec{X} - \vec{a})$,

analogous to results for sample mean/covariance:

$$\begin{aligned}
\vec{\mu}_Y &= \mathbb{E}\left(M(\vec{X} - \vec{a})\right) \\
&= M\left(\mathbb{E}(\vec{X}) - \vec{a}\right) \\
&= M\left(\vec{\mu}_X - \vec{a}\right)
\end{aligned}$$

$$\begin{aligned}
C_Y &= \mathbb{E}\left((M(\vec{X} - \vec{\mu}_X))(M(\vec{X} - \vec{\mu}_X))^T\right) \\
&= M\mathbb{E}\left((\vec{X} - \vec{\mu}_X)(\vec{X} - \vec{\mu}_X)^T\right)M^T \\
&= MC_XM^T
\end{aligned}$$

Special case: Sum of two RVs

Let $Z = X + Y$, or $Z = \vec{1}^T \begin{bmatrix} X \\ Y \end{bmatrix}$

$$\mu_Z = \mu_X + \mu_Y$$

$$\sigma_Z^2 = \sigma_X^2 + 2\sigma_{XY} + \sigma_Y^2$$

Special case: if X and Y are *independent*, then:

$\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$ and thus $\sigma_{XY} = 0$

$$\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2$$

$p_Z(z)$ is the *convolution* of $p_X(x)$ and $p_Y(y)$

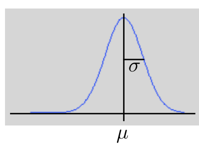
[on board]

Gaussian (a.k.a. "Normal") densities

One-dimensional:

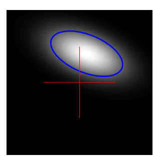
$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Alt. notation: $x \sim N(\mu, \sigma^2)$



Multi-dimensional:

$$p(\vec{x}) = \frac{1}{\sqrt{(2\pi)^N |C|}} e^{-(\vec{x}-\vec{\mu})^T C^{-1} (\vec{x}-\vec{\mu})/2}$$

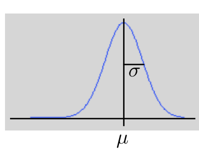


mean: [0.2, 0.8]
cov: [1.0 -0.3;
-0.3 0.4]

Gaussian properties

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

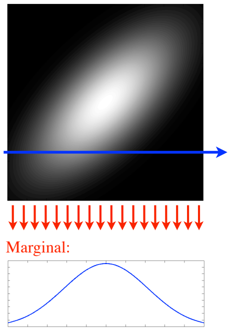
$$p(\vec{x}) = \frac{1}{\sqrt{(2\pi)^N |C|}} e^{-(\vec{x}-\vec{\mu})^T C^{-1} (\vec{x}-\vec{\mu})/2}$$



- joint density of indep Gaussian RVs is elliptical [easy]
- conditionals of a Gaussian are Gaussian [easy]
- marginals of a Gaussian are Gaussian [easy]
- product of two Gaussian dists is Gaussian [easy]
- sum of independent Gaussian RVs is Gaussian [moderate]
- the most random (max entropy) density of given variance [moderate]
- central limit theorem: sum of many indep. RVs is Gaussian [hard]

let $P = C^{-1}$ (the "precision" matrix)

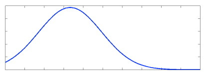
$$p(x_1|x_2 = a) \propto e^{-\frac{1}{2}\left[P_{11}(x_1-\mu_1)^2 + 2P_{12}(x_1-\mu_1)(a-\mu_2)+\ldots\right]}$$
$$= e^{-\frac{1}{2}\left[P_{11}x_1^2 + 2(P_{12}(a-\mu_2) - P_{11}\mu_1)x_1 + \ldots\right]}$$
$$= e^{-\frac{1}{2}\left(x_1-\mu_1+\frac{P_{12}}{P_{11}}(a-\mu_2)\right)P_{11}\left(x_1-\mu_1+\frac{P_{12}}{P_{11}}(a-\mu_2)\right)+\ldots}$$



Gaussian, with: $\mu = \mu_1 - \frac{P_{12}}{P_{11}}(a-\mu_2)$

$\sigma^2 = \frac{1}{P_{11}}$

Conditional:

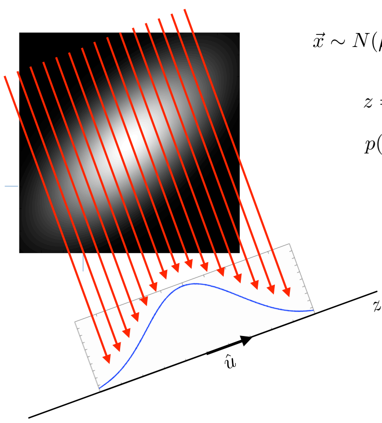


$$p(x_1) = \int p(\vec{x})\ dx_2$$

[on board]

Gaussian, with:
$$\mu = \mu_1$$
$$\sigma^2 = C_{11}$$

Generalized marginals of a Gaussian



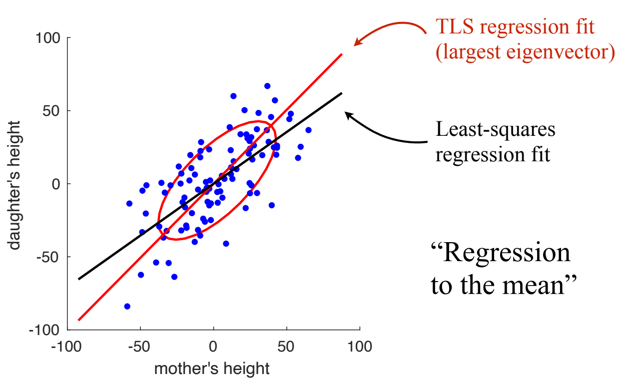
$\vec{x} \sim N(\vec{\mu}_x, C_x)$

$z = \hat{u}^T\vec{x}$

$p(z)$ is Gaussian, with:

$$\mu_z = \hat{u}^T\vec{\mu}_x$$
$$\sigma_z^2 = \hat{u}^T C_x \hat{u}$$

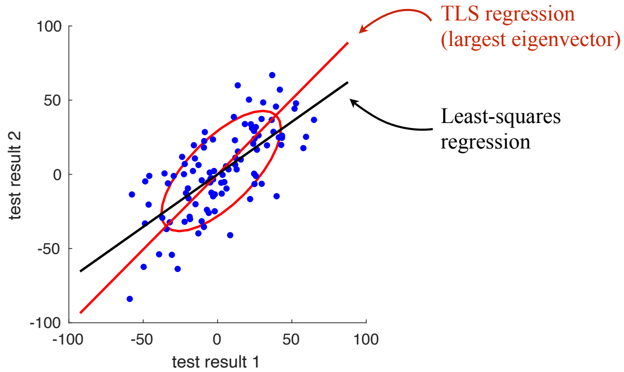
Correlation and regression



"Regression to the mean"

Francis Galton (1886). "Regression towards mediocrity in hereditary stature"

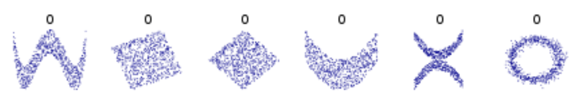
Correlation and regression



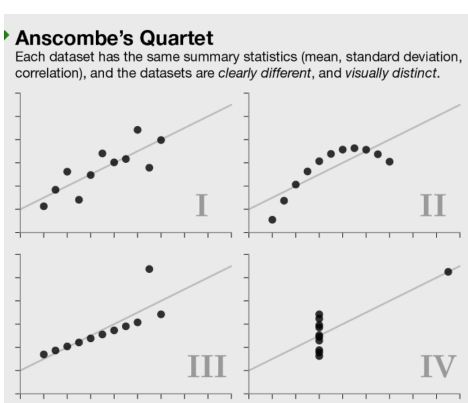
Correlation implies dependency

… but not slope

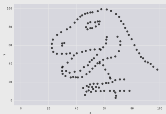
… and its absence does not imply independence!



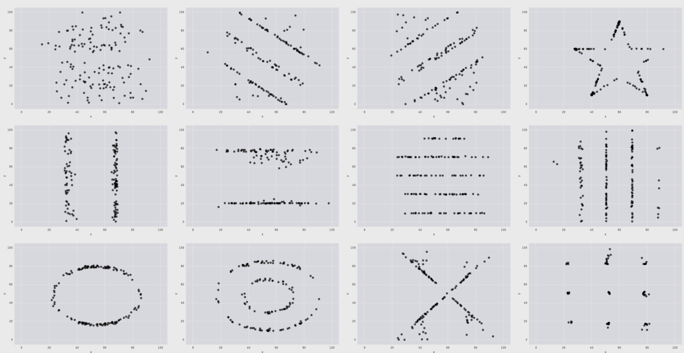
Correlation between variables does not uniquely indicate the shape of their joint distribution



More extreme examples !



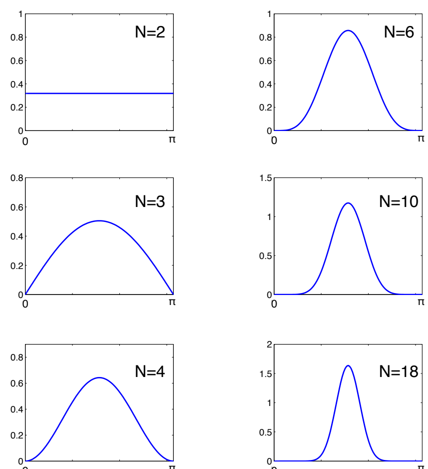
X Mean: 54.26
Y Mean: 47.83
X SD : 16.76
Y SD : 26.93
Corr. : -0.06



https://www.autodeskresearch.com/publications/samestats

Distribution of angles of pairs of random unit vectors

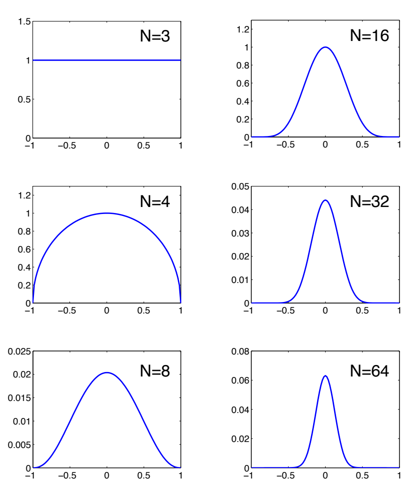
$$p(\theta) \propto \sin(\theta)^{(N-2)}$$



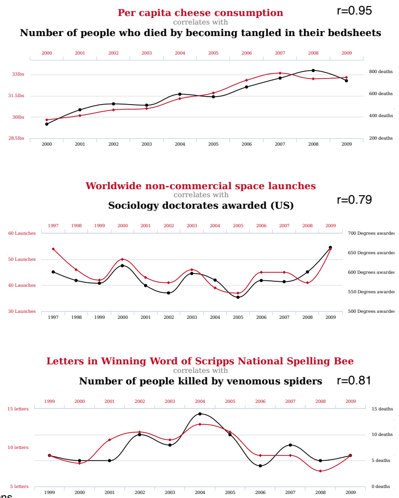
Lack of correlation is favored in N>3 dimensions

Null Hypothesis: Distribution of normalized dot product of pairs of Gaussian vectors in N dimensions:

$$(1-d^2)^{\frac{N-3}{2}}$$



Nevertheless, one can find correlation if one looks for it!



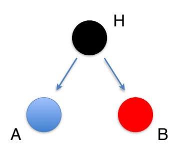
http://www.tylervigen.com/spurious-correlations

Covariation/correlation does not imply causation

- Correlation does not provide a direction for causality. For that, you need additional (temporal) information.
- More generally, correlations are often a result of hidden (unmeasured, uncontrolled) variables…

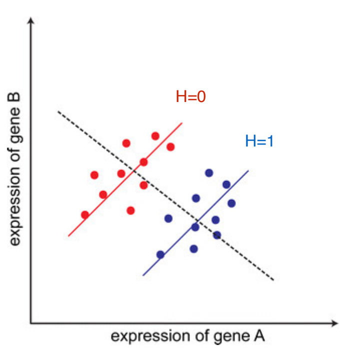
Example: conditional independence:

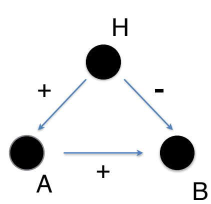
$$p(A, B \mid H) = p(A \mid H)p(B \mid H)$$



[On board: in Gaussian case, connections are explicit in the precision matrix]

Another example: "Simpson's paradox"

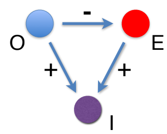




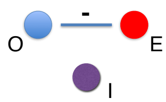
Milton Friedman's Thermostat

O = outside temperature (assumed cold)
I = inside temperature (ideally, constant)
E = energy used for heating

True interactions:



Observed statistics, $P=C^{-1}$:



Statistical observations:

- O and I uncorrelated
- I and E uncorrelated
- O and E anti-correlated

Some nonsensical conclusions:

- O and E have no effect on I, so shut off heater to save money!
- I is irrelevant, and can be ignored. Increases in E cause decreases in O.

Statistical summary cannot replace scientific reasoning/experiments!

Summary: Correlation misinterpretations

- Correlation implies dependency, but does *not* imply data lie near a line/plane/hyperplane.
- Independent implies uncorrelated. But uncorrelated does *not* imply independent.
- Correlation does *not* imply causation (and often arises from hidden common factors).
- Correlation is a **descriptive statistic**, and does not eliminate the need for reasoning/experiments/models!