Mathematical Tools
for Neural and Cognitive Science

Fall semester, 2019

Section 1: Linear Algebra

Linear Algebra

“Linear algebra has become as basic and as applicable as calculus, and fortunately it is easier”
- Gilbert Strang, *Linear Algebra and its Applications*

Vectors

In two or three dimensions, we can think these as arrows:

In higher dimensions, we typically use name to a "spike-plot"
Vector operations

- scalar multiplication
- addition, vector spaces
- length, unit vectors
- inner product (a.k.a. “dot” product)
  - properties: commutative, distributive
  - geometry: cosines, orthogonality test

[on board: geometry]

Inner product with a unit vector

- projection
- distance to line
- change of coordinates

[on board: geometry]

Vectors as “operators”

- “averager”
- “windowed averager”
- “gaussian averager”
- “local differencer”
- “component selector”

[on board]
Linear System

$S$ is a linear system if (and only if) it obeys the principle of superposition:

$S(ax + by) = aS(x) + bS(y)$

For any input vectors $\{x, y\}$ and any scalars $\{a, b\}$, the two diagrams at the right must produce the same response:

Linear Systems

- Very well understood (150+ years of effort)
- Excellent design/characterization toolbox
- An idealization (they do not exist!)
- Useful nevertheless:
  - conceptualize fundamental issues
  - provide baseline performance
  - good starting point for more complex models

Implications of Linearity
Implications of Linearity

Response to any input can be predicted from responses to impulses. This defines the operation of matrix multiplication.

Matrix multiplication

- Two interpretations of $M\vec{v}$ (see next slide):
  - input perspective: weighted sum of columns (from diagrams on previous slides)
  - output perspective: inner product with rows
- Distributive property (directly from linearity!)
- Associative property: cascade of two linear systems defines the product of two matrices
- Transpose $A^T$, symmetric matrices ($A = A^T$)
- Generally not commutative ($AB \neq BA$), but note that $(AB)^T = B^T A^T$
- Vectors: Inner products, Outer products (details on board)
Matrix multiplication

Two interpretations of $M \vec{v}$:

- **input perspective:** weighted sum of columns
- **output perspective:** dot product with rows

System response to first axis, $e_1$

Matrix multiplication: dimensional consistency

Orthogonal matrices
- square shape (dimensionality-preserving)
- rows are orthogonal unit vectors
- columns are orthogonal unit vectors
- performs a rotation of the vector space (with possible axis inversion)
- preserve vector lengths and angles (and thus, dot products)
- inverse is transpose

Diagonal matrices
- arbitrary rectangular shape
- all off-diagonal entries are zero
- squeeze/stretch along standard axes
- if non-square, creates/discards axes
- inverse is diagonal, with inverse of non-zero diagonal entries of original

All matrices

- Identity matrix (w/ axis flips)

Venn Diagram

Review/add:
1. two matrix types, each with geometric interpretation, understanding of inverse properties, "design" capabilities
2. geometry is about WHOLE SPACE
3. orthogonal: project onto orthogonal unit vectors => new coordinate system. Inverse: a) rotate back; b) multiply coordinates by unit vectors (expressed in original coordinate system).
4. diagonal: loss of dimensionality: smashing things down onto subspace.
Singular Value Decomposition (SVD)

- can express any matrix as \( M = U S V^T \)
  - "rotate, stretch, rotate"
  - columns of \( V \) are basis for input coordinate system
  - columns of \( U \) are basis for output coordinate system
  - \( S \) rescales axes, and determines what "gets through"
- interpretation: sum of "outer products"
- non-uniqueness? permutations, sign flips
- nullspace and rangespace
- inverse and pseudo-inverse

(details on board)
\[ M = \begin{bmatrix} U & S \\ \end{bmatrix} \]

orthogonal basis for “range space”

orthogonal basis for “null space”

orthogonal basis for “range space”

orthogonal basis for “null space”