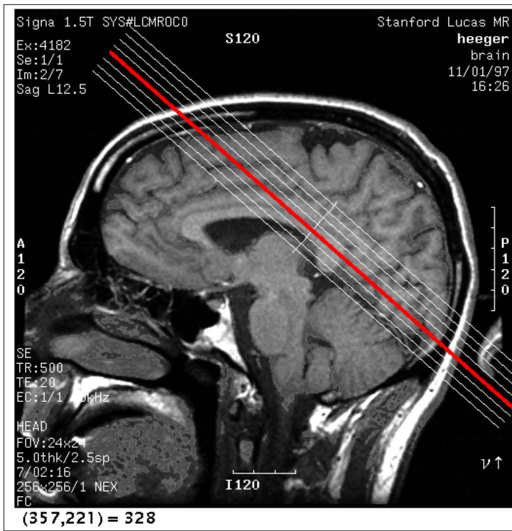
Mathematical Tools
for Neural and Cognitive Science

Fall semester, 2025

Section 6:

Decision-making
+
Categorization



Tumor, or not?

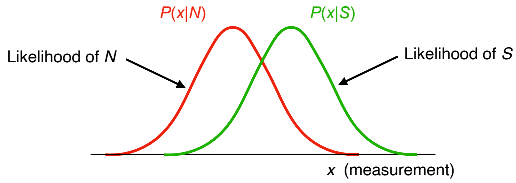
Decision-making and categorization (outline)

One-dimensional evidence and binary decision:
- Signal-detection theory
- Discriminability: Fisher Information

N-dimensional evidence and binary decision:
- Linear discriminant
- Quadratic discriminant

N-dimensional evidence and more than 2 categories
- Labeled data: ML or MAP extension of QDA
- Unlabeled data: K-means or soft K-means clustering

Signal Detection Theory (or, Statistical Decision Theory)

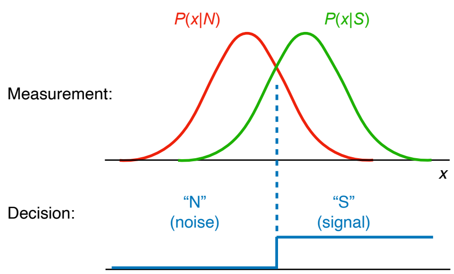


Stimulus is either "signal" (S) or "noise" (N).

$P(x \mid S)$ and $P(x \mid N)$ specify distributions of possible measurement x, conditioned on stimulus value.

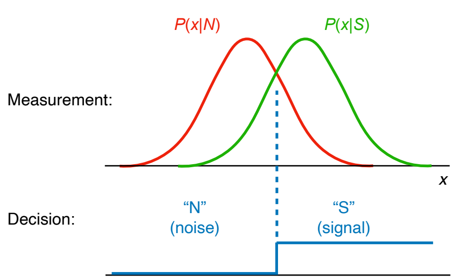
After x is measured, an ideal observer uses these as "likelihood functions" of the stimuli value (S or N).

Maximum likelihood (ML) decision rule



Say "yes" if $p(x \mid S) > p(x \mid N)$
"no" otherwise.

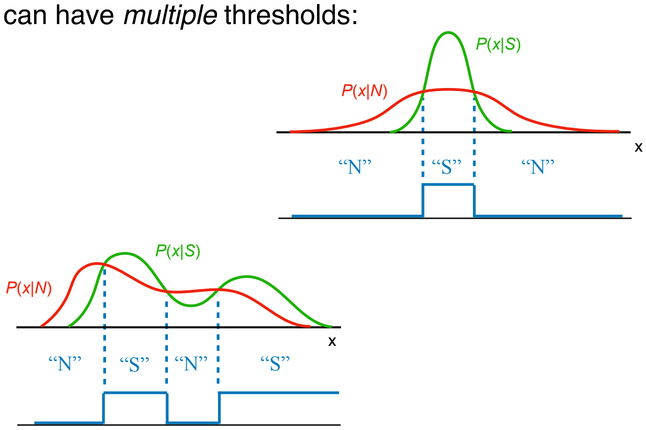
Maximum likelihood (ML) decision rule



Say "yes" if $x > \dfrac{\mu_S + \mu_N}{2} = c$
"no" otherwise.

(assuming equal-shaped symmetric unimodal distributions)

More generally, ML decision rule can have *multiple* thresholds:



Express posterior, using Bayes’ Rule

Likelihood

Posterior

Prior

$$P(S\,|\,x) = \frac{p(x\,|\,S)P(S)}{p(x)}$$

normalizing term

Maximum *a posteriori* (MAP) decision rule

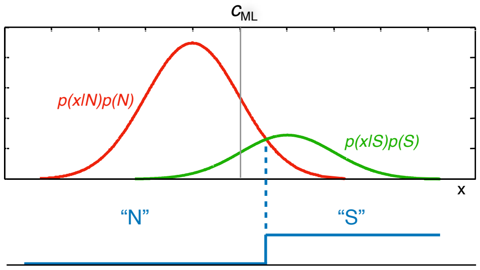
Say “yes” if $P(S\,|\,x) > P(N\,|\,x)$
“no” otherwise.

$\Rightarrow$ Say “yes” if $\dfrac{p(x\,|\,S)P(S)}{p(x)} > \dfrac{p(x\,|\,N)P(N)}{p(x)}$
“no” otherwise.

$\Rightarrow$ Say "yes” if $p(x\,|\,S)P(S) > p(x\,|\,N)P(N)$
“no” otherwise.

MAP decision rule

maximizes proportion of correct answers, *taking prior probability into account*.



Compared to ML threshold, the MAP threshold moves *away* from higher-probability option.

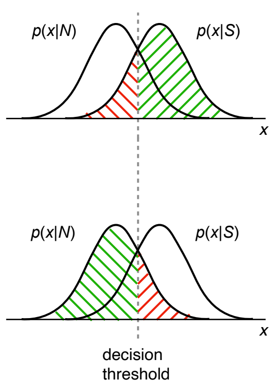
Ratio form of MAP decision rule

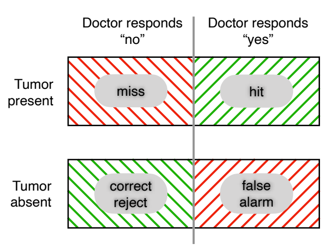
Say "yes" if $\dfrac{P(S|x)}{P(N|x)} > 1$

"no" otherwise, where

$$\underbrace{\frac{P(S|x)}{P(N|x)}}_{\text{"Posterior odds"}} = \underbrace{\left(\frac{p(x|S)}{p(x|N)}\right)}_{\text{"Likelihood ratio"}} \underbrace{\left(\frac{P(S)}{P(N)}\right)}_{\text{"Prior odds"}}$$

Signal Detection Theory: Potential outcomes





For threshold *t*, cumulatives *c()*

P(miss) = $c(t|S)$

P(hit) = $1-c(t|S)$

P(correct reject) = $c(t|N)$

P(false alarm) = $1-c(t|N)$

Bayes decision rule

("maximum expected gain" or "minimum Bayes risk")

Incorporate values for the four possible outcomes:

Payoff Matrix

Stimulus		Response: No	Response: Yes
	S	V_S^{No}	V_S^{Yes}
	N	V_N^{No}	V_N^{Yes}

Bayes Optimal Criterion

Stimulus		Response: No	Response: Yes
	S	V_S^{No}	V_S^{Yes}
	N	V_N^{No}	V_N^{Yes}

$$\mathbb{E}(Yes \mid x) = V_S^{Yes} P(S \mid x) + V_N^{Yes} P(N \mid x)$$
$$\mathbb{E}(No \mid x) = V_S^{No} P(S \mid x) + V_N^{No} P(N \mid x)$$

Say yes if $\mathbb{E}(Yes \mid x) \geq \mathbb{E}(No \mid x)$

Optimal Criterion

$$\mathbb{E}(Yes \mid x) = V_S^{Yes} P(S \mid x) + V_N^{Yes} P(N \mid x)$$
$$\mathbb{E}(No \mid x) = V_S^{No} P(S \mid x) + V_N^{No} P(N \mid x)$$

Say yes if $\mathbb{E}(Yes \mid x) \geq \mathbb{E}(No \mid x)$

Say yes if $\dfrac{P(S \mid x)}{P(N \mid x)} \geq \dfrac{V_N^{No} - V_N^{Yes}}{V_S^{Yes} - V_S^{No}} = \dfrac{V(\text{Correct} \mid N)}{V(\text{Correct} \mid S)}$

posterior odds

Apply Bayes' Rule

$$P(S\,|\,x) = \frac{p(x\,|\,S)P(S)}{p(x)}$$

$$P(N\,|\,x) = \frac{p(x\,|\,N)P(N)}{p(x)}$$

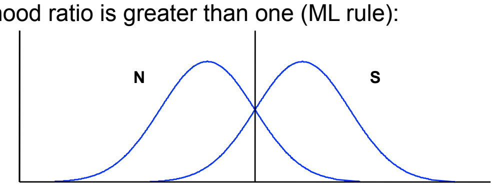
$$\underbrace{\frac{P(S\,|\,x)}{P(N\,|\,x)}}_{\text{Posterior odds}} = \underbrace{\left(\frac{p(x\,|\,S)}{p(x\,|\,N)}\right)}_{\text{Likelihood ratio}} \underbrace{\left(\frac{P(S)}{P(N)}\right)}_{\text{Prior odds}}$$

Optimal Criterion

Say yes if $\dfrac{P(S\,|\,x)}{P(N\,|\,x)} \geq \dfrac{V(\text{Correct}\,|\,N)}{V(\text{Correct}\,|\,S)}$

i.e., if $\dfrac{p(x\,|\,S)}{p(x\,|\,N)} \geq \dfrac{P(N)}{P(S)} \dfrac{V(\text{Correct}\,|\,N)}{V(\text{Correct}\,|\,S)} = \beta_{\text{opt}}$

Example, if equal priors and equal payoffs, say yes if the likelihood ratio is greater than one (ML rule):



Summary: nested optimal decision rules

(analogous to continuous case- see slides in prev section)

ML: Say "yes" if $\dfrac{p(x\,|\,S)}{p(x\,|\,N)} \geq 1$

MAP: Say "yes" if $\dfrac{p(x\,|\,S)}{p(x\,|\,N)} \geq \dfrac{P(N)}{P(S)}$

MEG: Say "yes" if $\dfrac{p(x\,|\,S)}{p(x\,|\,N)} \geq \dfrac{P(N)}{P(S)} \dfrac{V(\text{Correct}\,|\,N)}{V(\text{Correct}\,|\,S)}$

The likelihood ratio is a "sufficient statistic".

Standardized SDT

None of the derivations so far made any assumptions about the signal and noise distributions (even though the graphs looked Gaussian). Thus, all statements I've made about ML/MAP/MEG are true for *any* distributions: discrete (such as Poisson) vs. continuous, unequal signal vs. noise distributions, univariate vs. multivariate. The likelihood principle still holds.

However, the standard SDT model that is most often used assumes equal-variance Gaussians:

$$p(x|N) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu_N)^2}{2\sigma^2}\right) \text{ and } p(x|S) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu_S)^2}{2\sigma^2}\right)$$

Standardized SDT

$$p(x|N) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu_N)^2}{2\sigma^2}\right) \text{ and } p(x|S) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu_S)^2}{2\sigma^2}\right)$$

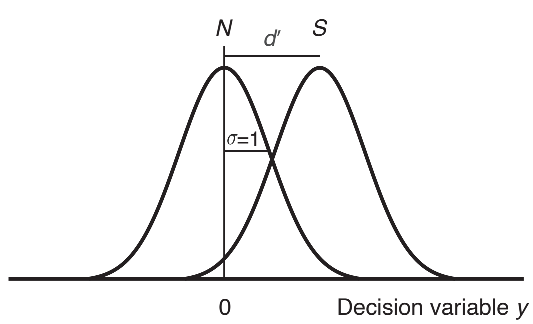
Change of variables: Let $y = \dfrac{x-\mu_N}{\sigma}$, $d' = \dfrac{\mu_S-\mu_N}{\sigma}$

$$d' = \frac{\text{separation}}{\text{width}}$$

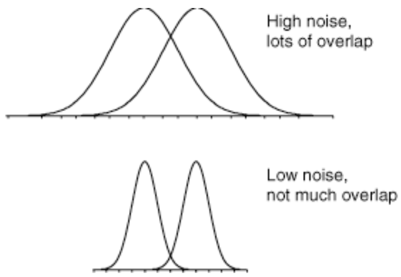
Then:

$$p(y|N) = \frac{1}{\sqrt{2\pi}}\exp\frac{-y^2}{2} \text{ and } p(y|S) = \frac{1}{\sqrt{2\pi}}\exp\frac{-(y-d')^2}{2}$$

Standardized SDT



Signal Detection Theory: discriminability (d')

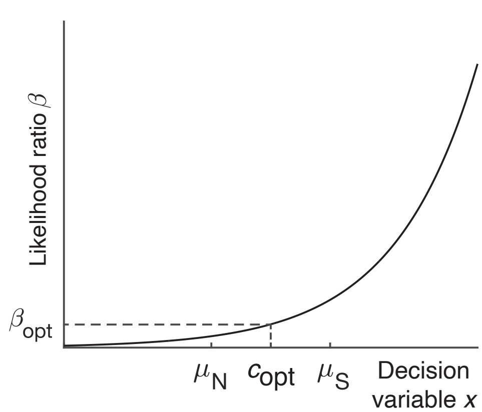


Standardized SDT

Likelihood ratio for $y = c$ is:

$$\frac{p(c \mid S)}{p(c \mid N)} = \frac{\exp\left[-\frac{(c-d')^2}{2}\right]}{\exp\left[-\frac{c^2}{2}\right]} = \exp\left[cd' - \frac{d'^2}{2}\right]$$

Standardized SDT



Standardized SDT

Likelihood ratio for $y = c$ is:

$$\beta_{\text{opt}} = \frac{p(c_{\text{opt}} \mid S)}{p(c_{\text{opt}} \mid N)} = \frac{\exp\left[-\frac{(c_{\text{opt}} - d')^2}{2}\right]}{\exp\left[-\frac{c_{\text{opt}}^2}{2}\right]} = \exp\left[c_{\text{opt}} d' - \frac{d'^2}{2}\right]$$

$$\log \beta_{\text{opt}} = c_{\text{opt}} d' - \frac{d'^2}{2}$$

$$\begin{aligned} c_{\text{opt}} &= \frac{d'}{2} + \frac{\log \beta_{\text{opt}}}{d'} \\ &= \frac{d'}{2} + \frac{1}{d'}\left[\log \frac{P(N)}{P(S)} + \log \frac{V(\text{Correct} \mid N)}{V(\text{Correct} \mid S)}\right] \end{aligned}$$

Standardized SDT

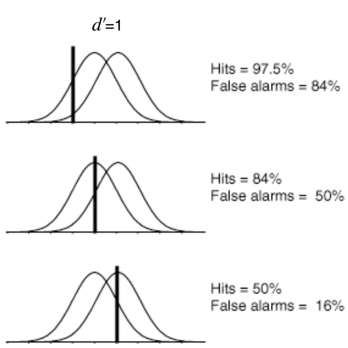
$$c_{\text{opt}} = \frac{d'}{2} + \frac{1}{d'}\left[\log \frac{P(N)}{P(S)} + \log \frac{V(\text{Correct} \mid N)}{V(\text{Correct} \mid S)}\right]$$

Optimal criterion is the ML criterion, shifted by a term that is a function of the prior odds plus a term that is a function of the payoff ratio.

Note that this additivity of the effects of priors and payoffs is *not* seen in human behavior.

Locke, S. M., Gaffin-Cahn, E., Hosseiniaveh, N., Mamassian, P. & Landy, M. S. (2020). *Attention, Perception, & Psychophysics*, *82*, 3158-3175.

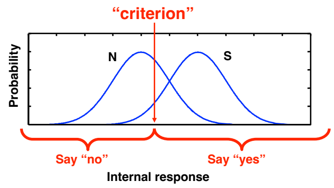
Signal Detection Theory: Criterion



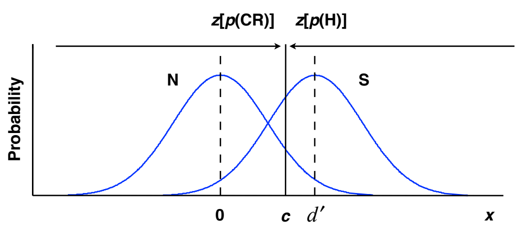
Example applications of SDT

- Vision
  - Detection (something vs. nothing)
  - Discrimination (lower vs greater level of: intensity, contrast, depth, slant, size, frequency, loudness, ...
- Memory (internal response = trace strength = familiarity)
- Neurometric function/discrimination by neurons (internal response = spike count)

From experimental measurements, assuming Gaussian distributions, can we determine the underlying values of d' and "criterion" (threshold)?



SDT: Estimating d' and c



$$\begin{aligned} d' &= z[P(\text{Hit})] + z[P(\text{Correct Reject})] \\ &= z[P(\text{Hit})] - z[P(\text{False Alarm})] \\ c &= z[P(\text{Correct Reject})], \text{ where} \end{aligned}$$

$$z(P) = \Phi^{-1}(P), \text{ where } \Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp \frac{z^2}{2} dz$$

SDT: Estimating d' and c

$$\begin{aligned} d' &= z[P(\text{Hit})] + z[P(\text{Correct Reject})] \\ &= z[P(\text{Hit})] - z[P(\text{False Alarm})] \\ c &= z[P(\text{Correct Reject})] \end{aligned}$$

Stimulus	Response: No	Response: Yes
S	n_{Miss}	n_{Hit}
N	n_{CR}	n_{FA}

$$\hat{P}_{\text{Hit}} = n_{\text{Hit}}/(n_{\text{Hit}} + n_{\text{Miss}})$$
$$\hat{P}_{\text{FA}} = n_{\text{FA}}/(n_{\text{FA}} + n_{\text{CR}})$$

SDT: Estimating d' and c

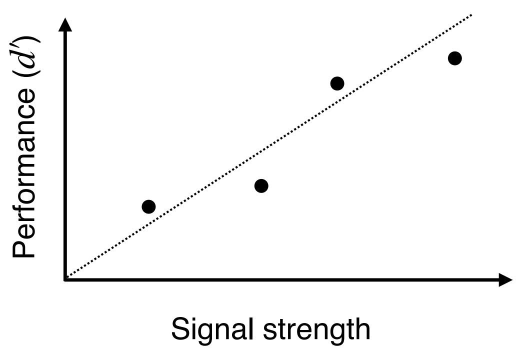
$$d' = z[P(\text{Hit})] + z[P(\text{Correct Reject})]$$
$$= z[P(\text{Hit})] - z[P(\text{False Alarm})]$$
$$c = z[P(\text{Correct Reject})]$$

Stimulus	Response: No	Response: Yes
S	$n_{\text{Miss}} + 0.5$	$n_{\text{Hit}} + 0.5$
N	$n_{\text{CR}} + 0.5$	$n_{\text{FA}} + 0.5$

$$\hat{P}_{\text{Hit}} = (n_{\text{Hit}} + 0.5)/(n_{\text{Hit}} + n_{\text{Miss}} + 1)$$
$$\hat{P}_{\text{FA}} = (n_{\text{FA}} + 0.5)/(n_{\text{FA}} + n_{\text{CR}} + 1)$$

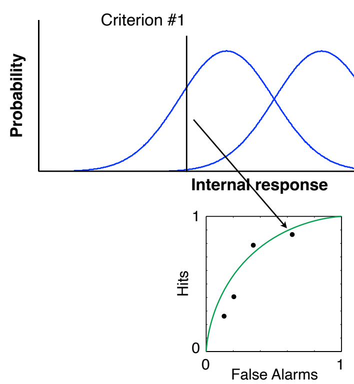
Hautus, M. J. (1995). Corrections for extreme proportions and the biasing effects on estimated values of d'. *Behavior Research Methods, Instruments, & Computers*, *27*, 46-51.

SDT: Psychometric function



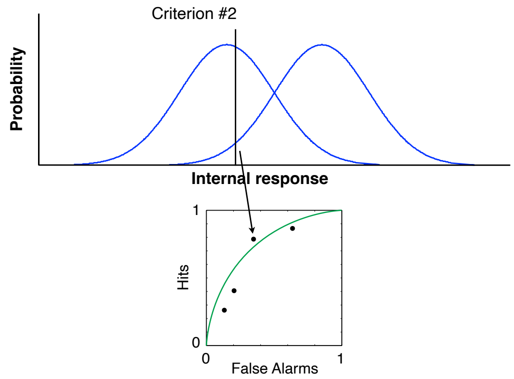
Note: 4 hit rates and one, shared, false-alarm rate

ROC (Receiver Operating Characteristic)

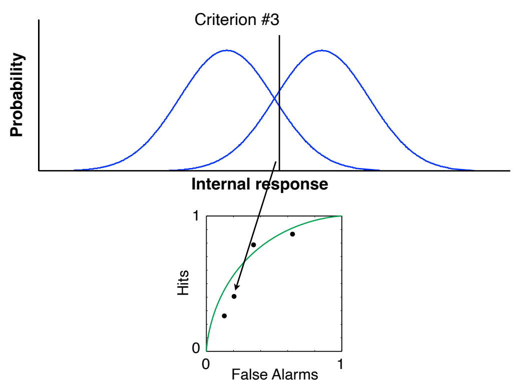


Plot anti-cumulatives:
1 - c(*t* | N) vs. *1-c*(*t* | *S*)
as threshold *t* varies

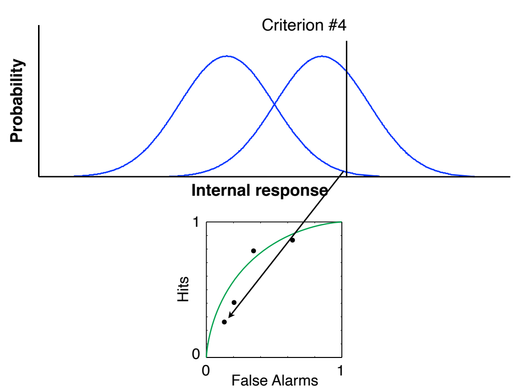
ROC (Receiver Operating Characteristic)



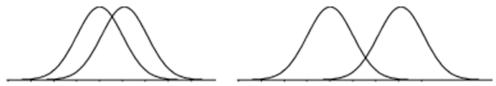
ROC (Receiver Operating Characteristic)



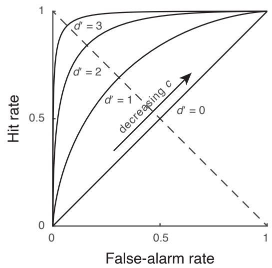
ROC (Receiver Operating Characteristic)



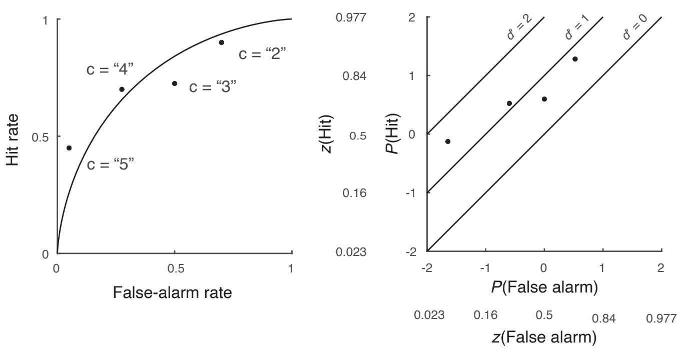
ROC (Receiver Operating Characteristic)



$d' = 1$ (lots of overlap) $d' = 3$ (less overlap)



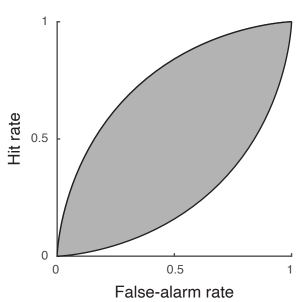
ROC (Receiver Operating Characteristic)



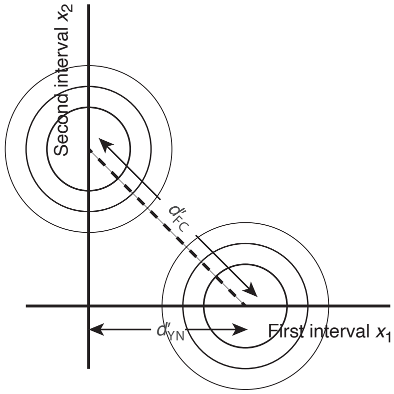
Or do a ML fit: Dorfman, D. D., & Alf, J., E. (1969). Maximum likelihood estimation of parameters of signal detection theory and determination of confidence intervals: rating-method data. *Journal of Mathematical Psychology*, *6*, 487-496.

ROC (Receiver Operating Characteristic)

Achievable performance



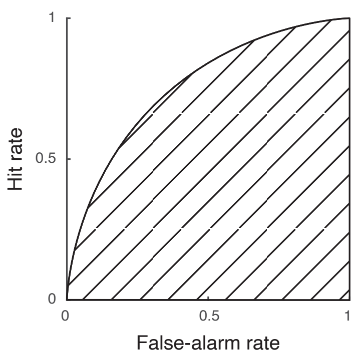
SDT and 2AFC



Yeshurun, Y., Carrasco, M., & Maloney, L. T. (2008). Bias and sensitivity in two-interval forced choice procedures: Tests of the difference model. *Vision Research*, *48*, 1837-1851.

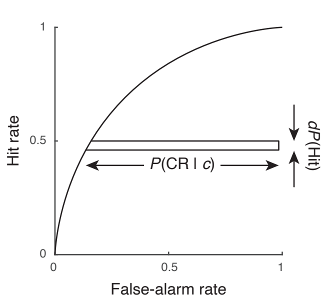
Area under the ROC

Area under curve = %correct in a 2AFC task



Area under the ROC

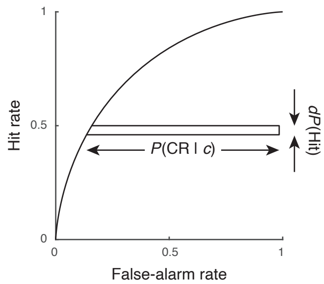
Area under curve = %correct in a 2AFC task

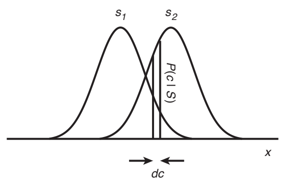


$$\text{AUROC} = \int_0^1 P(\text{Correct reject} \mid \text{criterion } c)\, dP(\text{Hit} \mid \text{criterion } c)$$

Area under the ROC

Area under curve = %correct in a 2AFC task





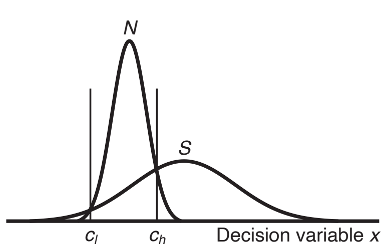
Slope of the ROC = likelihood ratio!

Area under the ROC

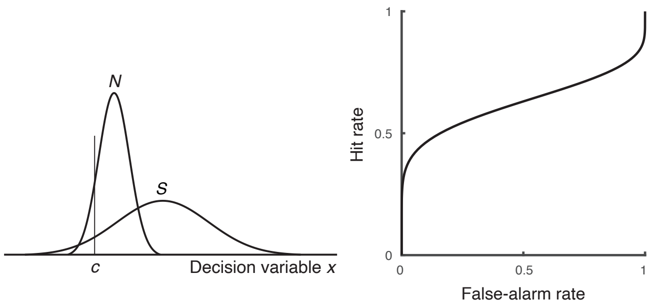
Area under curve = %correct in a 2AFC task

$$\begin{aligned}\text{AUROC} &= \int_0^1 P(\text{Correct reject} \,|\, \text{criterion } c)\, dP(\text{Hit} \,|\, \text{criterion } c) \\ &= \int_{-\infty}^{\infty} p(x < c \,|\, N)\, p(\text{measurement is } c \,|\, S)\, dc \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{c} p(x \,|\, N)\, p(\text{measurement is } c \,|\, S)\, dx dc \\ &= \int_{-\infty}^{\infty} p(\text{measurement is } c \,|\, S) \int_{-\infty}^{c} p(x \,|\, N)\, dx dc \\ &= P_{\text{FC}}.\end{aligned}$$

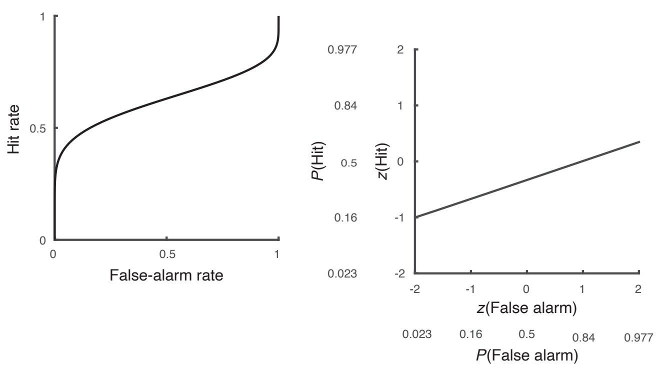
SDT with unequal variances



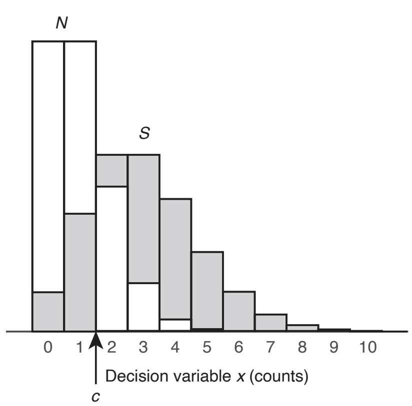
SDT with unequal variances



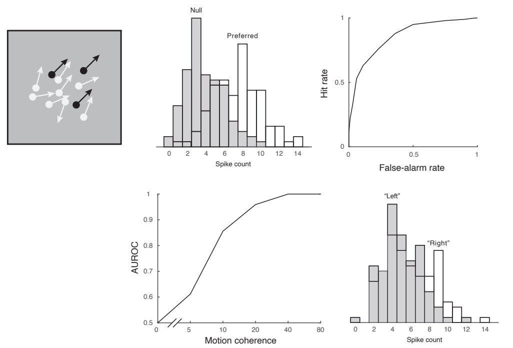
SDT with unequal variances



SDT with a discrete (Poisson) distribution



Area under the ROC - Poisson case or with data: Neurometric function and choice probability



Britten, K. H., Shadlen, M. N., Newsome, W. T., & Movshon, J. A. (1992). The analysis of visual motion: a comparison of neuronal and psychophysical performance. *Journal of Neuroscience*, *12*, 4745-4765.
Britten, K. H., Newsome, W. T., Shadlen, M. N., Celebrini, S. & Movshon, J. A. (1996). A relationship between behavioral choice and the visual responses of neurons in macaque MT. *Visual Neuroscience*, *13*, 87-100.

Decision-making and categorization

One-dimensional evidence and binary decision:
- Signal-detection theory
- Discriminability: Fisher Information

N-dimensional evidence and binary decision:
- Linear discriminant
- QDA

N-dimensional evidence and more than 2 categories
- Labeled data: ML or MAP extension of QDA
- Unlabeled data: K-means or soft K-means clustering

Fisher Information

- Second-order expansion of the (expected) negative log likelihood:

$$I(s) = -\mathbb{E}\left[\frac{\partial^2 \log p(r|s)}{\partial s^2}\right]$$

- Provides a bound on "precision" of unbiased estimators: (the "Cramér-Rao bound") $\sigma^2(s) \geq \dfrac{1}{I(s)}$

- Perceptually, provides a bound on **discriminability**: (Series et. al. 2009) $D(s) \leq \sqrt{I(s)}$

- Examples: with mean stimulus response $\mu(s)$

Gaussian case: $p(r|s) \sim \mathcal{N}(\mu(s), \sigma^2)$ $\qquad I(s) = [\mu'(s)]^2/\sigma^2$

Poisson case: $p(r|s) \sim \text{Poiss}(\mu(s))$ $\qquad I(s) = [\mu'(s)]^2/\mu(s)$

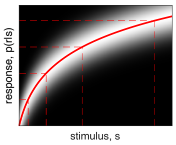
Example: Weber's law

[Weber, 1834]

$$D(s) \propto \frac{1}{s}$$ (discrimination thresholds proportional to stimulus strength)

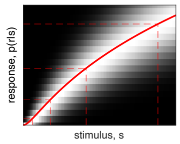
Assuming $I(s) \propto \frac{1}{s^2}$ what internal representation can explain this? Many!

additive Gaussian noise, with mean $\mu(s) = \log(s) + c$



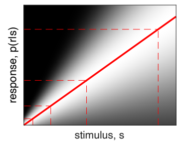
entirely due to response mean [Fechner, 1860]

Poisson noise, with mean $\mu(s) = [\log(s) + c]^2$



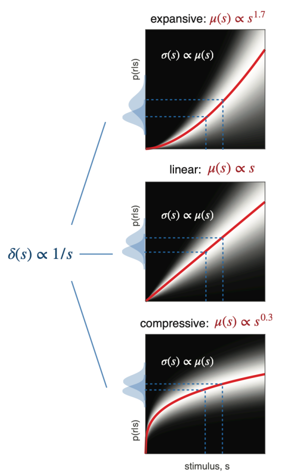
discrete representation, depends on both mean and variance

multiplicative Gaussian noise, with mean $\mu(s) = \alpha s$



entirely due to response variance

S.S. Stevens. "To Honor Fechner and Repeal His Law: A power function, not a log function, describes the operating characteristic of a sensory system" (1961)



[Zhou, Duong & EPS, 2022]

Three examples with different power-law mean response, each consistent with Weber's law discriminability.

Decision-making and categorization

One-dimensional evidence and binary decision:
- Signal-detection theory
- Discriminability: Fisher Information

N-dimensional evidence and binary decision:
- Linear discriminant
- QDA

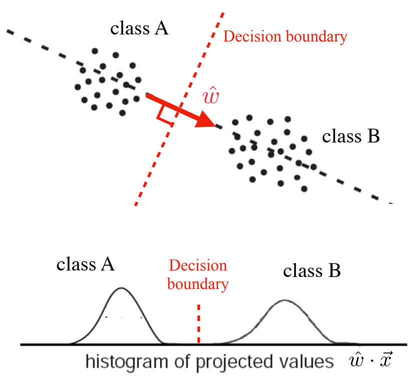
N-dimensional evidence and more than 2 categories
- Labeled data: ML or MAP extension of QDA
- Unlabeled data: K-means or soft K-means clustering

Decision/classification in multiple dimensions

- Data-driven linear classifiers:
  - Prototype Classifier - minimize distance to class mean
  - Fisher Linear Discriminant (FLD) - maximize d'
  - Support Vector Machine (SVM) - maximize margin
- Statistical:
  - ML/MAP/Bayes under a probabilistic model
  - e.g.: Gaussian, identity covariance (same as Prototype)
  - e.g.: Gaussian, equal covariance (same as FLD)
  - e.g.: Gaussian, general case (Quadratic Discriminator)
- Some Examples:
  - Visual gender classification
  - Neural population decoding

Linear Classifier

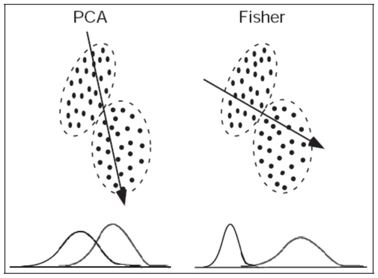
Find unit vector $\hat{w}$ ("discriminant") that best separates the distributions



Simplest linear discriminant: the Prototype Classifier

$$\hat{w} = \frac{\vec{\mu}_A - \vec{\mu}_B}{\|\vec{\mu}_A - \vec{\mu}_B\|}$$

Fisher Linear Discriminant



$$\max_{\hat{w}} \frac{\left[\hat{w}^T(\vec{u}_A - \vec{u}_B)\right]^2}{\left[\hat{w}^T C_A \hat{w} + \hat{w}^T C_B \hat{w}\right]}$$ (note: this is d' squared!)

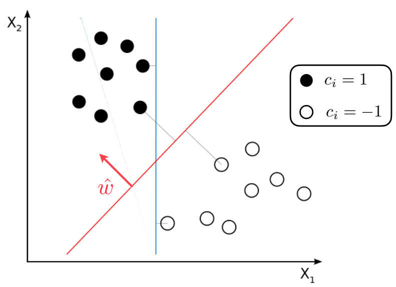
optimum: $\hat{w} = C^{-1}(\vec{u}_A - \vec{u}_B)$, where $C = \frac{1}{2}(C_A + C_B)$

Support Vector Machine (SVM)

(widely used in machine learning, but no closed form solution)

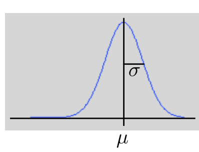
Maximize the "margin" (gap between data sets):

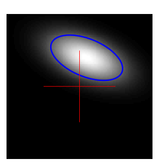
find largest m, and $\{\hat{w}, b\}$ s.t. $c_i(\hat{w}^T \vec{x}_i - b) \geq m, \quad \forall\, i$



Reminder: Multi-D Gaussian densities

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



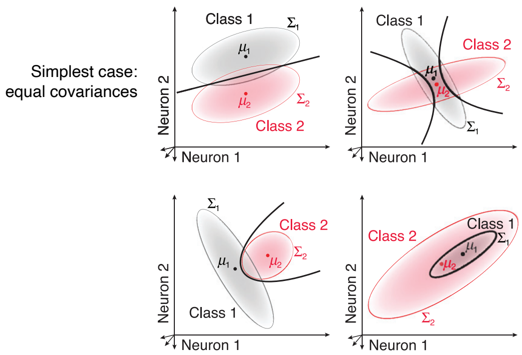


mean: $[0.2, 0.8]$
cov: [1.0 -0.3;
-0.3 0.4]

$$p(\vec{x}) = \frac{1}{\sqrt{(2\pi)^N |C|}}\, e^{-(\vec{x}-\vec{\mu})^T C^{-1} (\vec{x}-\vec{\mu})/2}$$

ML (or MAP) classifier for two Gaussians

Decision boundary is *quadratic*, with four possible geometries:



[on board]

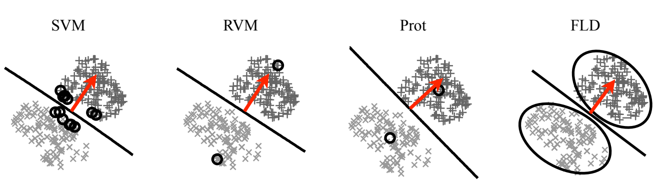
[figure: Pagan et al. 2016]

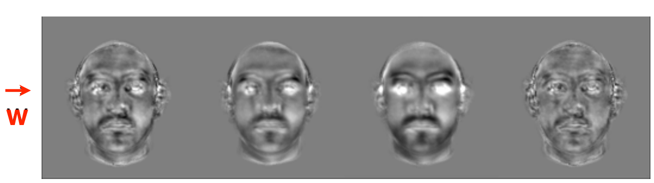
A perceptual example: Biological gender identification (XX vs. XY)

- 200 face images (100 male, 100 female)
- Adjusted for position, size, intensity/contrast
- Labeled by 27 human subjects

[Graf & Wichmann, NIPS*03]

Linear classifiers

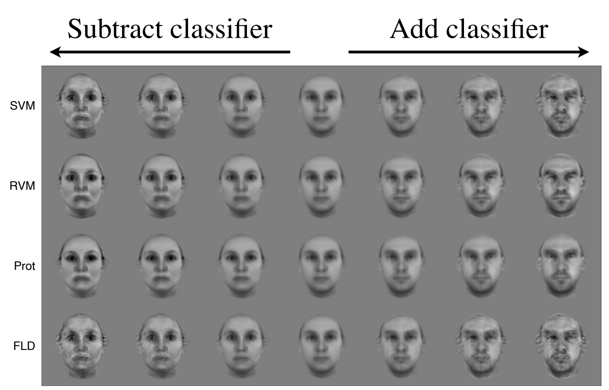




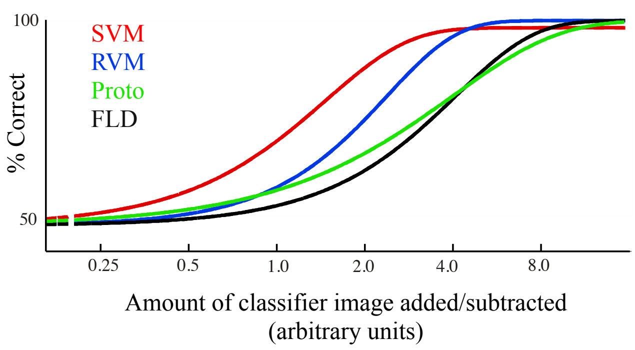
Four linear classifiers, trained on human data

Model validation/testing

- Cross-validation: Subject responses [% correct, reaction time, confidence] are explained
  - very well by SVM
  - moderately well by RVM / FLD
  - not so well by Prot
- Do these decision "models"make testable predictions? Synthesize optimally discriminable faces...



[Wichmann, et. al; NIPS*04]



[Wichmann, et. al; NIPS*04]

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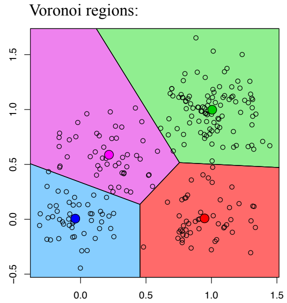
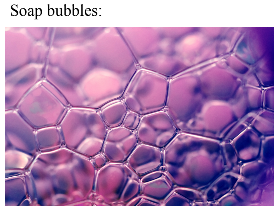
More than two categories, labeled data

If means and covariances are known, and the covariances are circular and identical across categories, this reduces to selecting the nearest neighbor:

Voronoi regions:

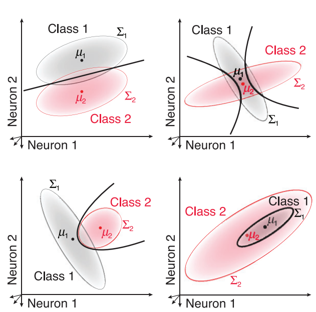


Soap bubbles:



More two categories, labeled data, Gaussian distributions, but not necessarily circular nor equal across categories.

ML (or MAP) classifier generalizes QDA:



[figure: Pagan et al. 2016]

Unlabeled data: Clustering

- K-Means (Lloyd, 1957)

- "Soft-assignment" version of K-means
  (a form of Expectation-Maximization - EM)

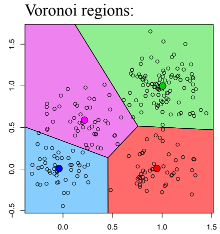
- In general, alternate between:
  1) Estimating cluster assignments (classification)
  2) Estimating cluster parameters
- Coordinate descent: converges to (possibly local) minimum
- Need to choose K (number of clusters) - cross-validation!

K-Means clustering algorithm

Alternate between two steps:

1. Estimate cluster assignments: given class centers, assign each point to closest one:

Voronoi regions:

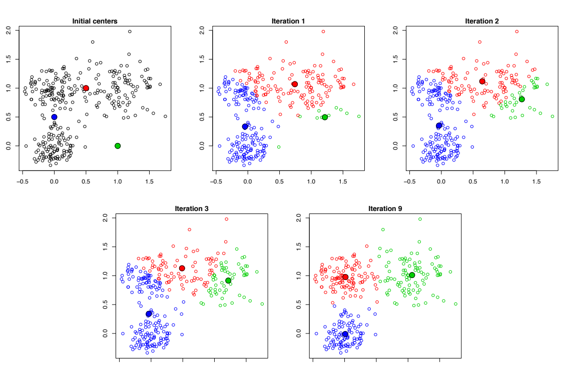


Soap bubbles:

2. Estimating cluster parameters: given assignments, re-estimate the centroid of each cluster.

K-means example

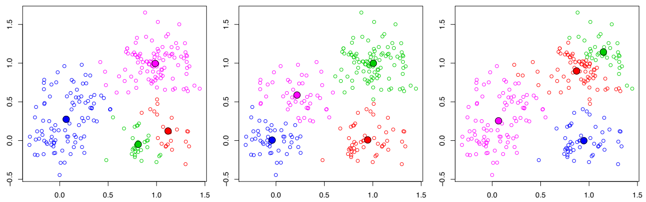
$N = 300$, and $K = 3$



[from R. Tibshirani, 2013]

K-means optimization failures

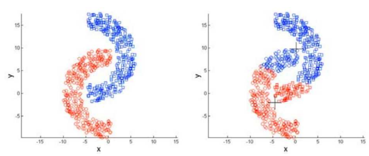
Initialization matters (due to local minima) …
Three solutions obtained with different random starting points:



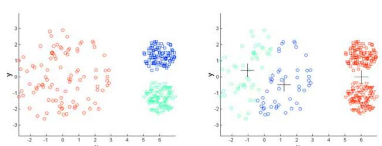
[from R. Tibshirani, 2013]

K-means systematic failures

Non-convex/non-round-shaped clusters



Clusters with different densities



Picture courtesy: Christof Monz (Queen Mary, Univ. of London)

ML for discrete mixture of Gaussians: soft K-means

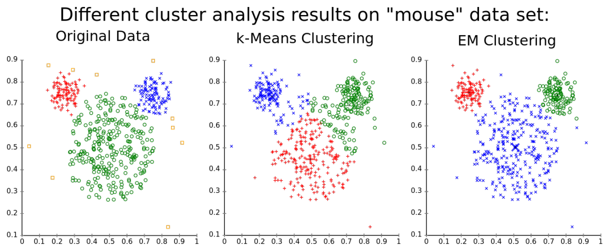
$$p(\vec{x}_n | a_{nk}, \vec{\mu}_k, \Lambda_k) \propto \sum_k \frac{a_{nk}}{\sqrt{|\Lambda_k|}} e^{-(\vec{x}_n - \vec{\mu}_k)^T \Lambda_k^{-1} (\vec{x}_n - \vec{\mu}_k)/2}$$

a_{nk} = assignment *probability*

$\{\vec{\mu}_k, \Lambda_k\}$ = mean/covariance of class *k*

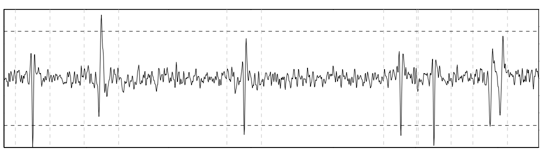
Intuition: alternate between maximizing these two sets of variables ("coordinate descent")

Essentially, a version of K-means with "soft" (i.e., continuous, as opposed to binary) assignments!



[wikipedia]

Application to neural “spike sorting”

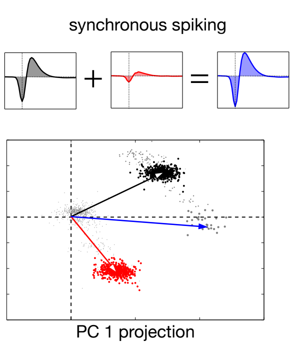


Standard solution:

1. Threshold to find segments containing spikes
2. Reduce dimensionality of segments using PCA
3. Identify spikes using clustering (e.g., K-means)

Note: Fails for overlapping spikes!

Failures of clustering for near-synchronous spikes

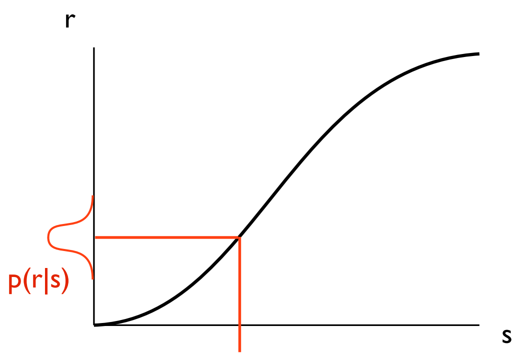


[Pillow et. al. 2013]

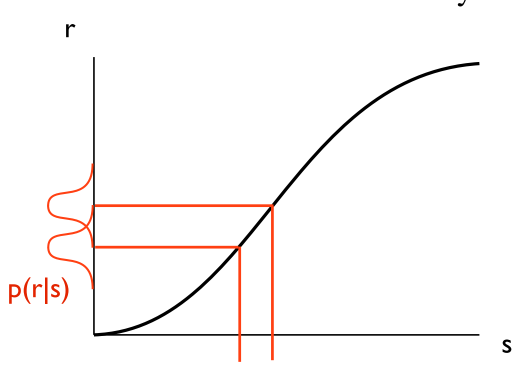
“Decoding” neural populations?

- Connecting neural response to behavior
- Engineering: Brain-Computer Interfaces
- Test/compare encoding models

Encoding determines discriminability

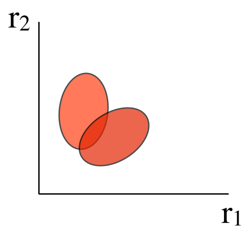


Probabilistic encoding model determines discriminability



Discriminability (d’) is (approximately) slope/stdev

Two neurons



Same fundamental issues as 1D case:

- Probabilistic encoding determines discriminability
- Intuitively, overlap is distance/spread
- For linear decoding: project onto discrimination axis

I. Simple/intuitive population decoding

- Linear? $\hat{s}(\vec{r}) = \sum_n r_n s_n$

  (simple, but usually doesn't work well)

- Winner-take-all $\hat{s}(\vec{r}) = s_m, \qquad m = \arg\max_n \{r_n\}$

  (simple, but discontinuous and noise-susceptible)

- Population vector [Georgopoulos et.al., 1986] $\hat{s}(\vec{r}) = \dfrac{\sum_n r_n s_n}{\sum_n r_n}$

  (also simple, more robust)

II. Statistically optimal decoding

- Maximum likelihood (ML) $\hat{s}(\vec{r}) = \arg\max_s p(\vec{r}|s)$
- Maximum a posteriori (MAP) $\hat{s}(\vec{r}) = \arg\max_s \; p(\vec{r}|s) \cdot p(s)$
- Minimum Mean Squared Error (MMSE), a.k.a. Bayes Least Squares (BLS) $\hat{s}(\vec{r}) = \mathbf{E}(s|\vec{r})$

ML decoding for a Poisson-spiking neural population

[Ma, Beck, Latham, Pouget, 2006; Jazayeri & Movshon, 2006]

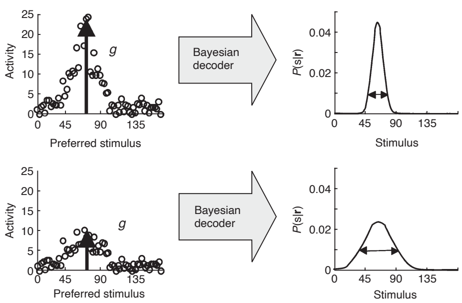
$$p(\vec{r}|s) = \prod_{n=1}^{N} \frac{h_n(s)^{r_n}\ e^{-h_n(s)}}{r_n!}$$

$$\log\left(p(\vec{r}|s)\right) = \sum_{n=1}^{N} r_n \log(h_n(s)) - h_n(s) - \log(r_n!)$$

If $\sum_{n=1}^{N} h_n(s)$ is constant (i.e., tuning curves "tile"), just minimize the response-weighted sum of log tuning curves.

Special cases allow closed-form solutions:

- Gaussian tuning curves $\quad h_n(s) = \exp\left(-(s-s_n)^2/2\sigma^2\right)$
- von Mises tuning curves $\quad h_n(s) = \exp\left(\kappa\cos(s-s_n)\right)$

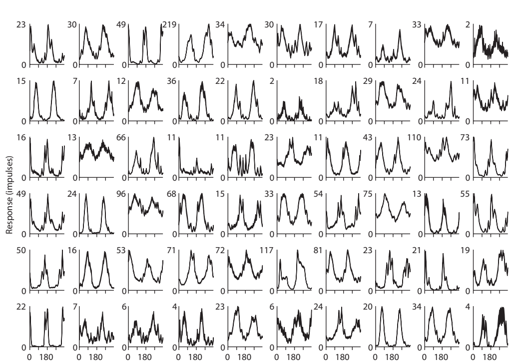


$$\hat{s}(\vec{r}) = \frac{\sum_n r_n s_n}{\sum_n r_n} \qquad \hat{\sigma}(\vec{r}) = \frac{\sigma_{TC}}{\sqrt{\sum r_n}}$$

[Ma, Beck, Latham & Pouget, 06]

Population Decoding

The data: tuning curves f_i



[Graf, Kohn, Jazayeri & Movshon, 11]

Comparing population decoders

1) The ML decoder, assuming independent Poisson responses (the PID):

$$\log L(\theta) = \log\left(\prod_{i=1}^{N} p(r_i|\theta)\right) = \sum_{i=1}^{N} \log\left(\frac{f_i(\theta)^{r_i}}{r_i!}\exp(-f_i(\theta))\right)$$

$$= \sum_{i=1}^{N} \log(f_i(\theta))r_i - \sum_{i=1}^{N} f_i(\theta) - \sum_{i=1}^{N} \log(r_i!) = \sum_{i=1}^{N} W_i(\theta)r_i + B(\theta)$$

For discrimination between two values, likelihood ratio is *linear* function of responses:

$$\log LR(\theta_1, \theta_2) = \log\left(\frac{L(\theta_1)}{L(\theta_2)}\right) = \log L(\theta_1) - \log L(\theta_2)$$

$$= \sum_{i=1}^{N} [W_i(\theta_1) - W_i(\theta_2)]r_i + [B(\theta_1) - B(\theta_2)]$$

$$= \sum_{i=1}^{N} w_i(\theta_1, \theta_2)r_i + b(\theta_1, \theta_2)$$

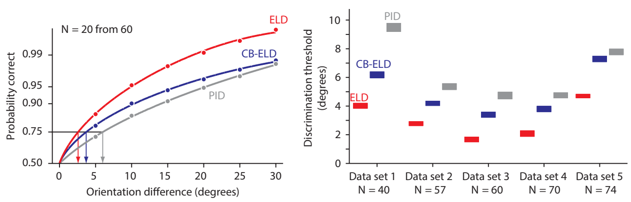
[Graf, Kohn, Jazayeri & Movshon, 11]

Comparing population decoders

2) Alternatively, compute an SVM on the measured response vectors for each orientation, the empirical linear decoder (ELD):

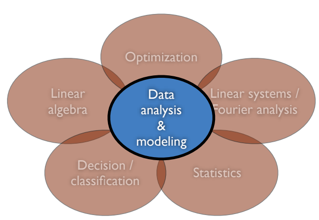
$$y(\theta_1, \theta_2) = \sum_{i=1}^{N} w_i(\theta_1, \theta_2)r_i + b(\theta_1, \theta_2)$$

3) For each neuron and orientation, shuffle the responses across trials and train a new SVM, the correlation-blind empirical linear decoder (CB-ELD).

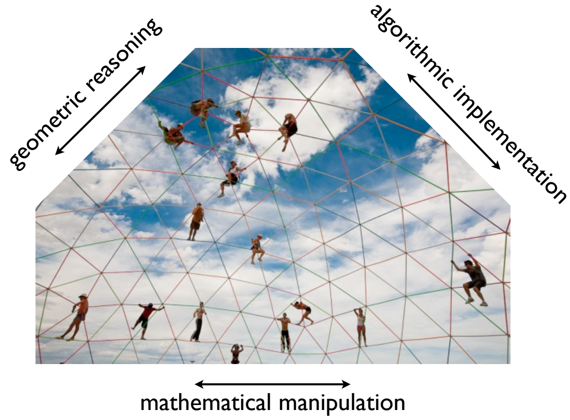


[Graf, Kohn, Jazayeri & Movshon, 11]

Where we've been…



- Linear algebra / linear systems
  - Ex: Trichromacy
- Least squares
  - regression / PCA
- Linear shift-invariant systems
  - convolution / Fourier transforms
  - Ex: Auditory filtering
- Summary statistics - dispersion, central tendency
- Statistical inference
  - estimation, bias, variance, convergence
  - maximum likelihood estimator (MLE), MAP, Bayes
  - Ex: signal detection theory
  - Classification, clustering
- Model fitting
  - model comparison, overfitting, regularization, cross-validation
  - Ex: fitting an LNP model
  - Ex: population decoding


geometric reasoning
algorithmic implementation
mathematical manipulation