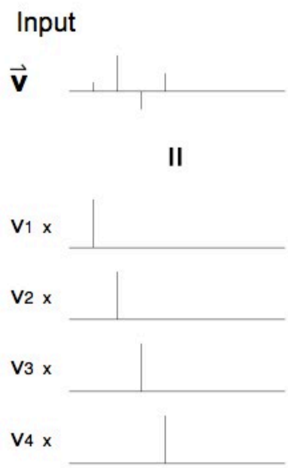
Linearity = Superposition



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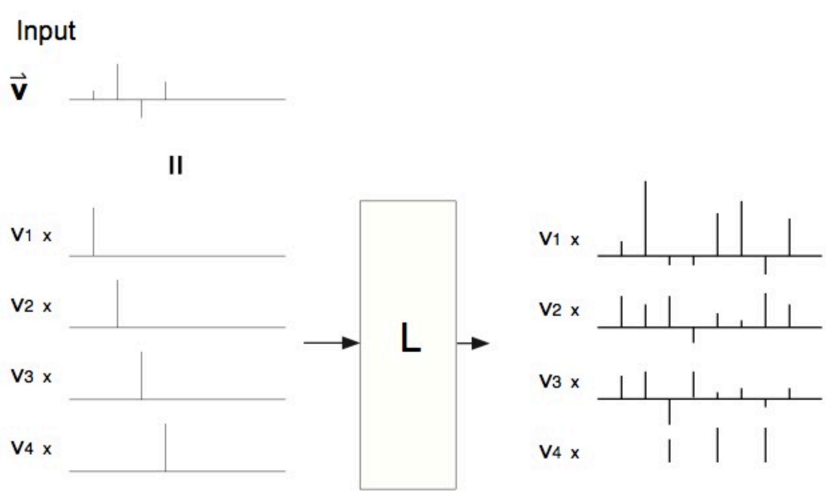

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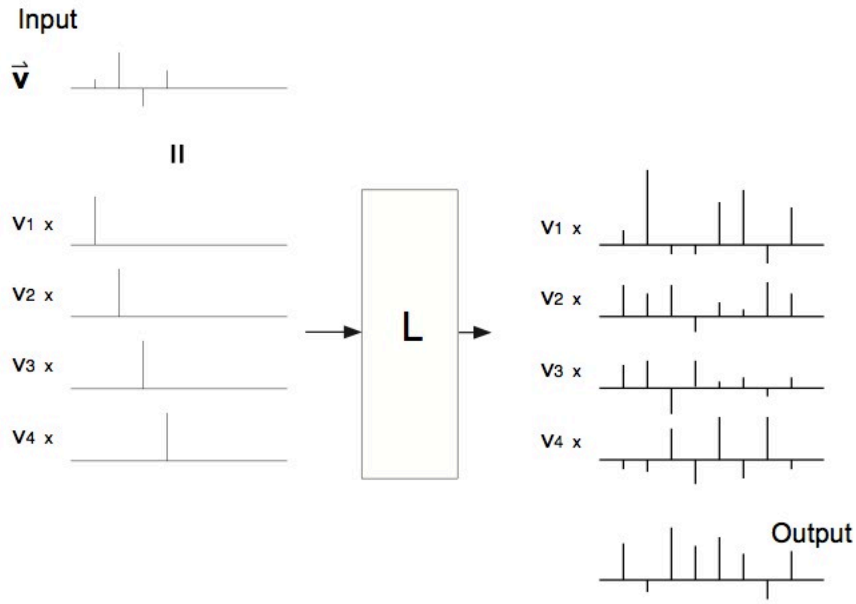

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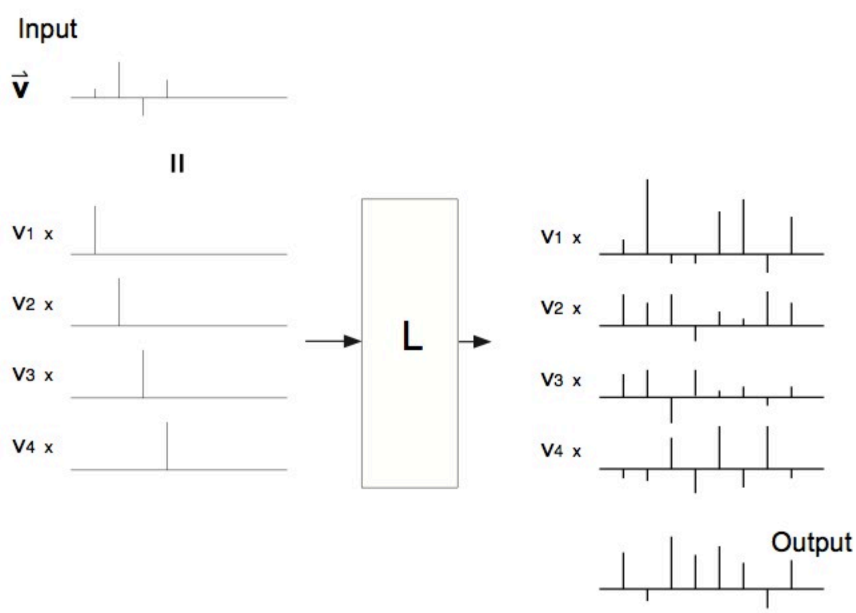
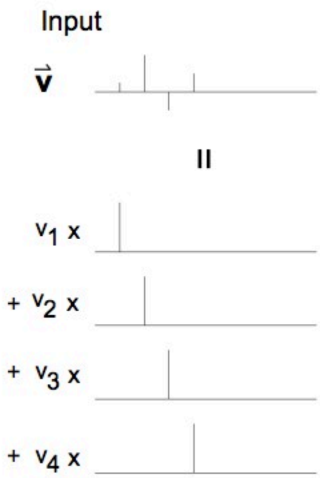


Finite linear systems correspond to matrix multiplication

Linear shift-invariant (LSI) system



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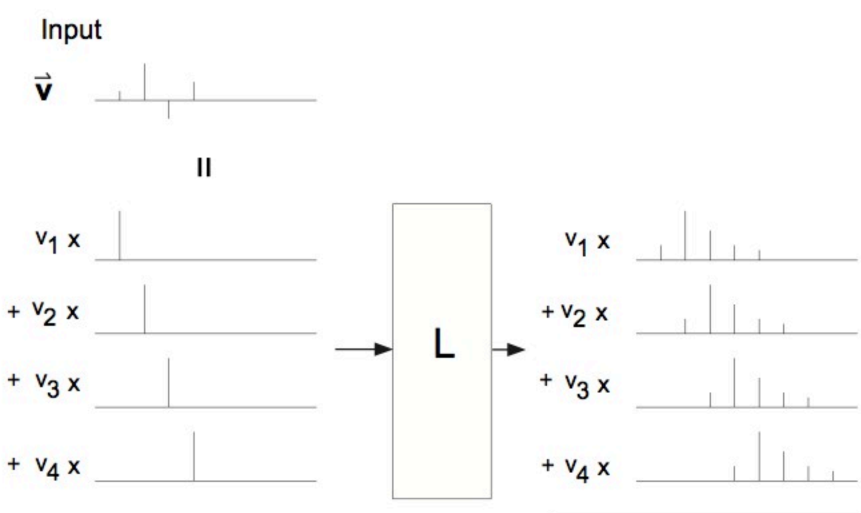

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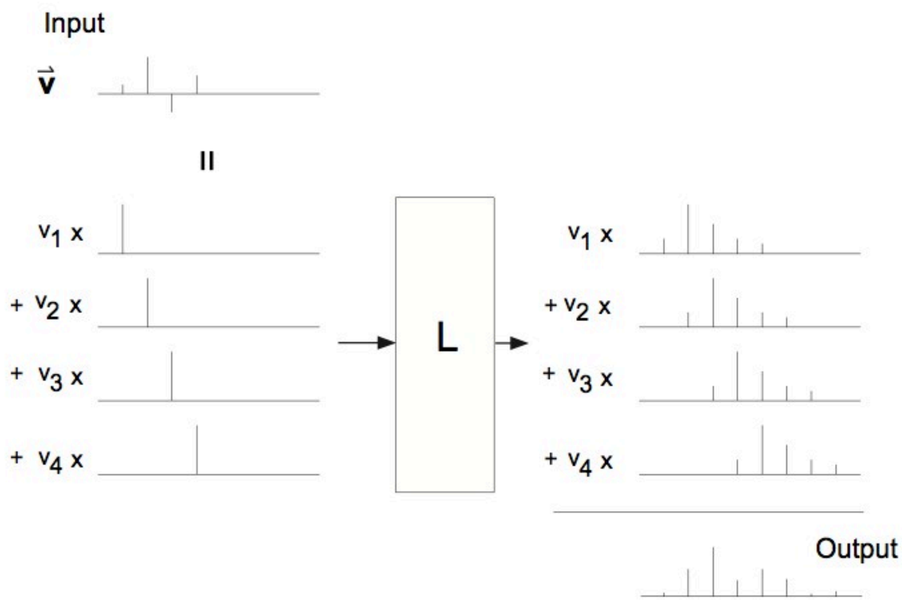

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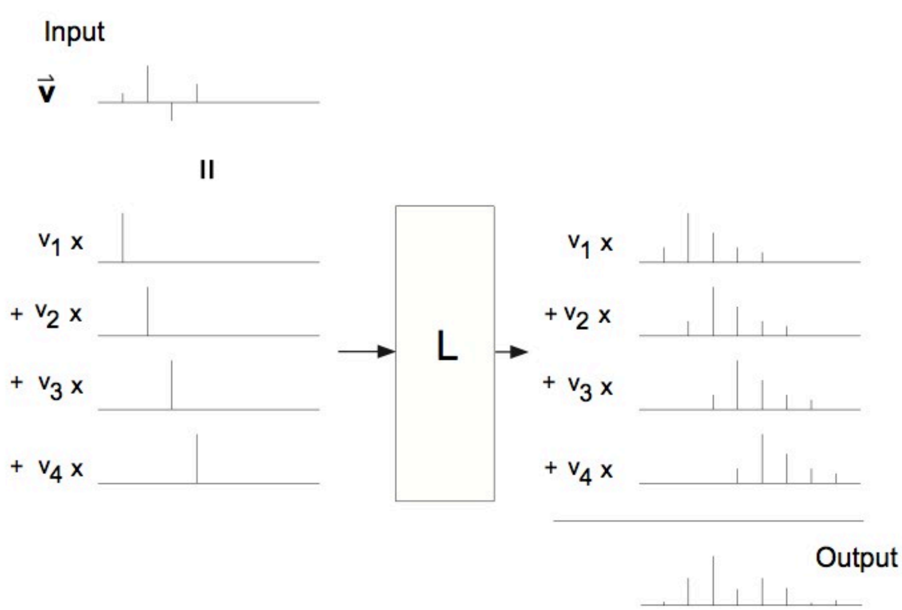
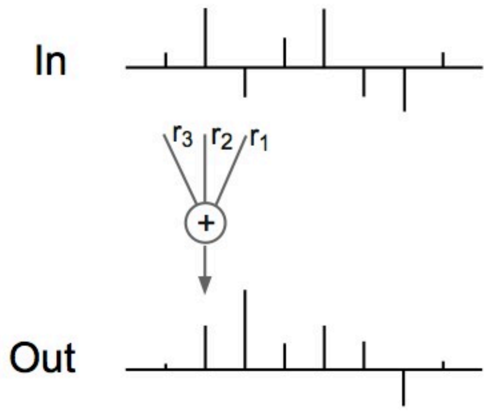


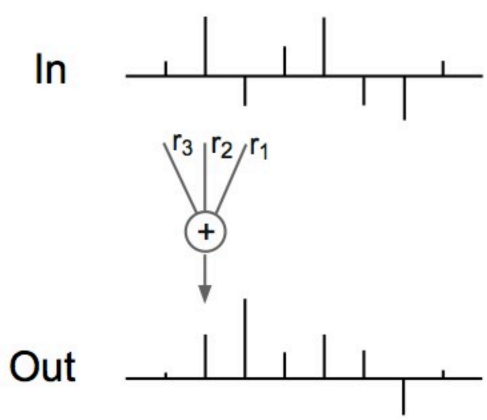
LSI systems characterized by impulse response

Convolution



- matrix view
- boundaries
- examples: impulse, delay, average, difference

Convolution



$$y(n) = \sum_k r(n-k)x(k) = \sum_k r(k)x(n-k)$$

- matrix view
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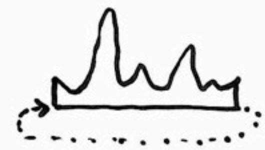
Boundaries (1D)

- Circular (toroidal)

  $I(\vec{n}) = I(\vec{n} \bmod \vec{N})$



- Zeros

  $I(\vec{n}) = 0$

- Reflect

  $I(\vec{n}) = I(-n), \; n<0$

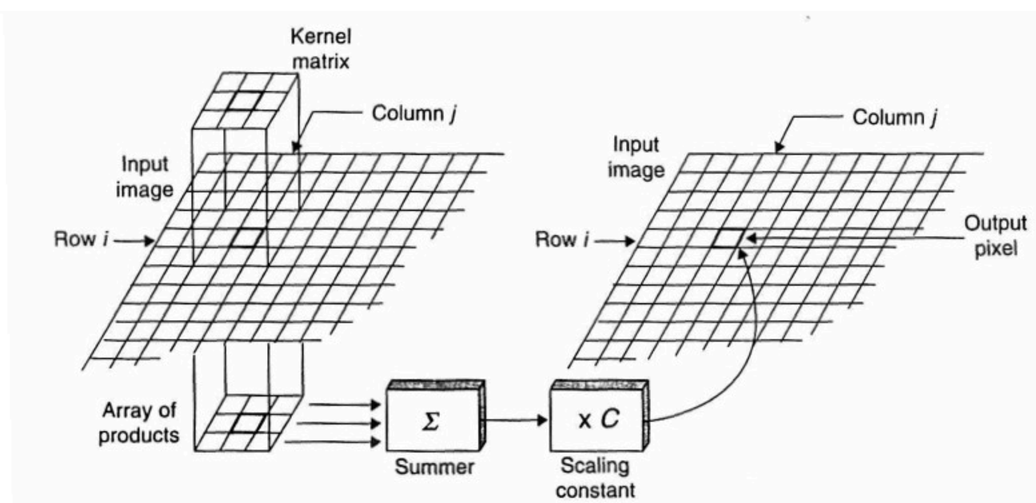
  $I(n) = I(2N-n), \; n>N$

- Extend

  $I(n) = 2I(0) - I(-n), \; n<0$

  $I(n) = 2I(N) - I(2N-n), \; n>N$

2D convolution (FIR)

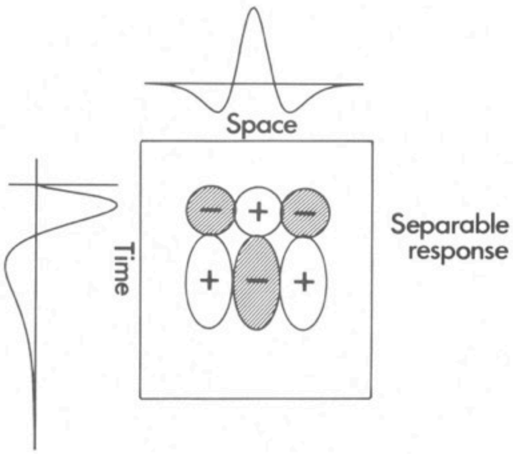


Indexing: annoying, but important... where is the "center" of the filter?

[figure: Castleman]

2D convolution

- sliding window
- boundaries
- separable filters



Discrete Sinusoids

$$\cos(\omega n), \qquad \omega = 2\pi k/N$$

Discrete Sinusoids

"frequency" (cycles/vectorLength)

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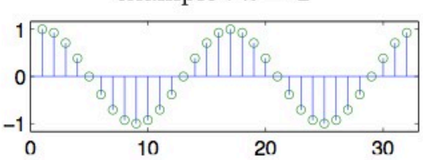
example : $k = 2$



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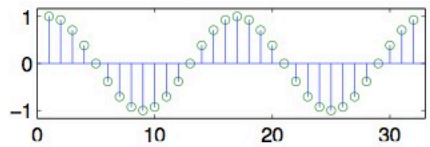

$\cos(\omega n), \qquad \omega = 2\pi k / N$

"frequency" (radians/sample)

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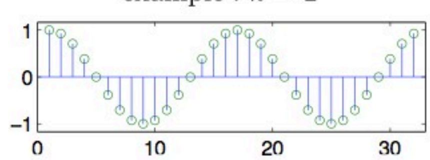

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More generally: $A \cos(\omega n - \phi)$

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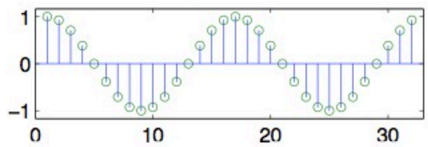

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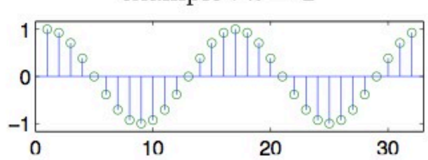
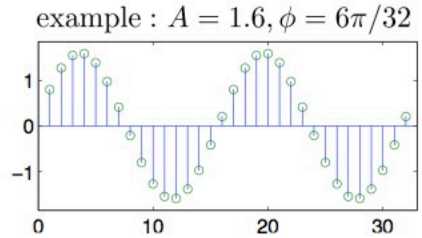


More generally: $A\cos(\omega n - \phi)$

"amplitude"

"phase" (radians)

example : $A = 1.6, \phi = 6\pi/32$



Shifting Sinusoids

$$A\cos(\omega n - \phi) = A\cos(\phi)\cos(\omega n) + A\sin(\phi)\sin(\omega n)$$

fixed cos/sin vectors:

A *shifted* sinusoidal vector can be written as a weighted sum of two *fixed* sinusoidal vectors!

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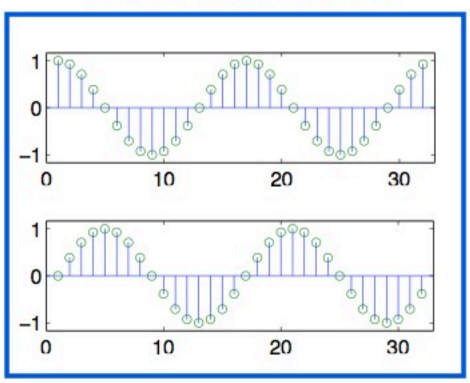

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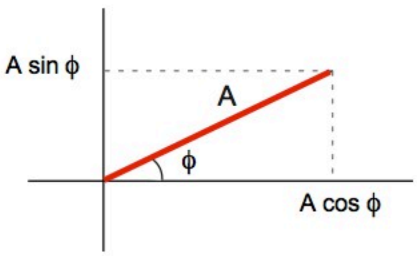

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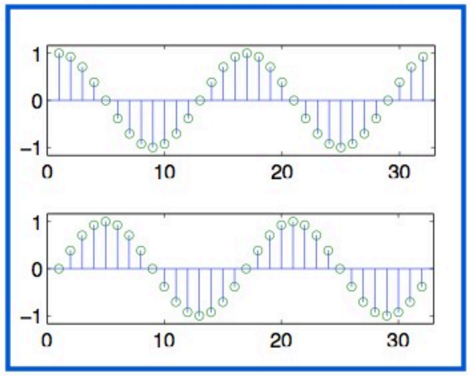

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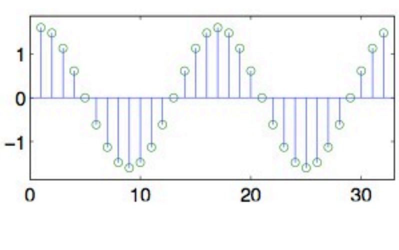
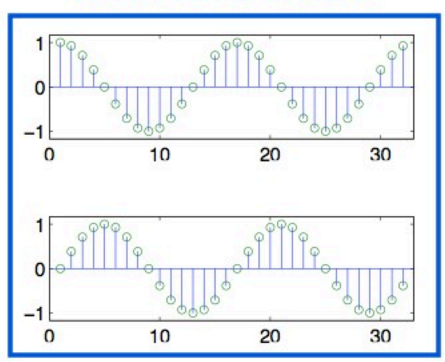


A = 1.6, ϕ = 2π0/12



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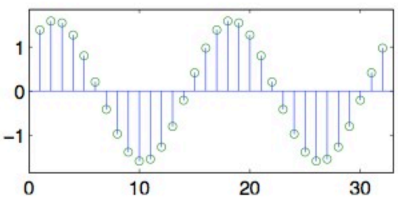

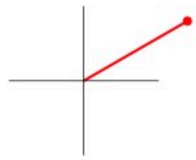

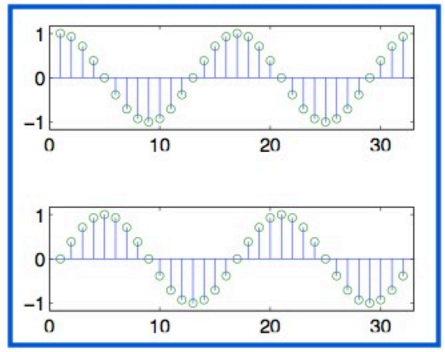

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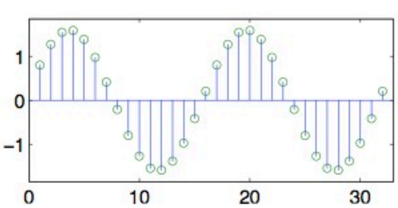

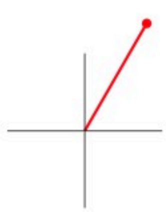

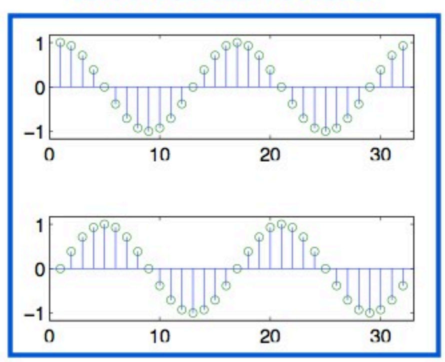

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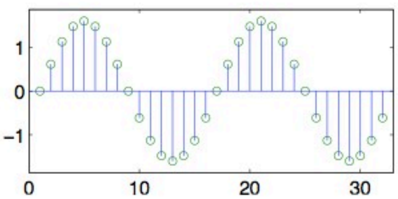

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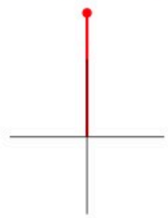

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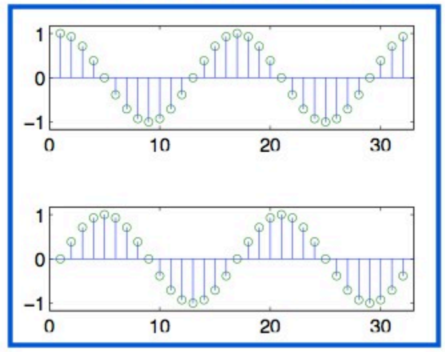

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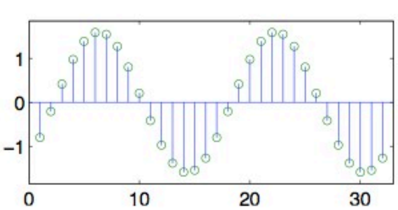

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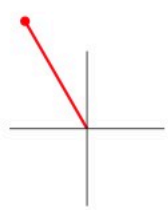

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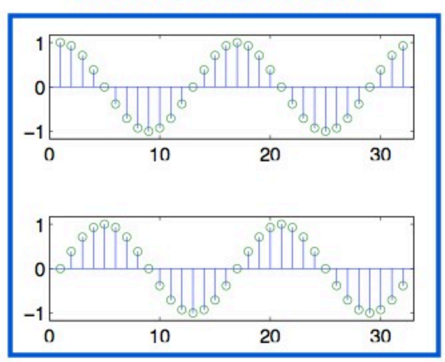

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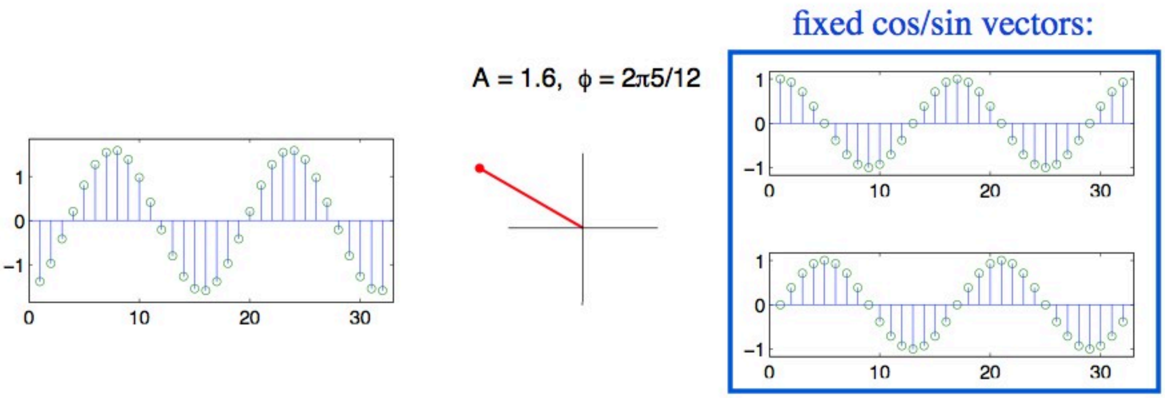

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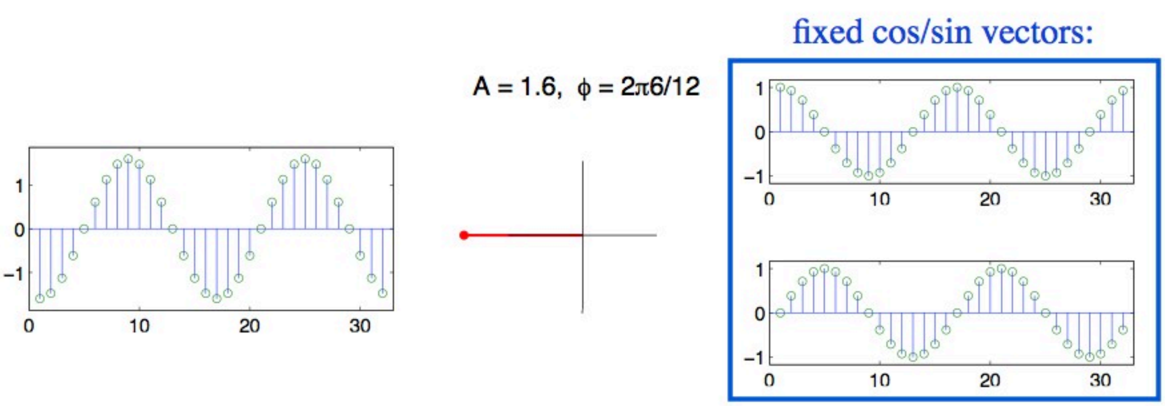

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$$y(n) = \sum_m r(m) \cos(\omega(n-m)) \qquad \text{(convolution)}$$

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\end{aligned}
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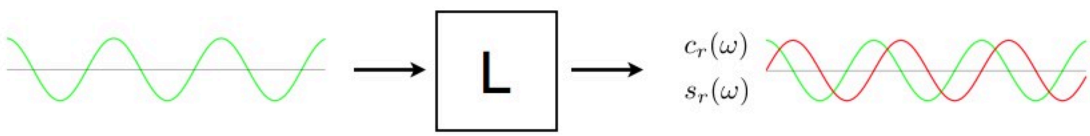

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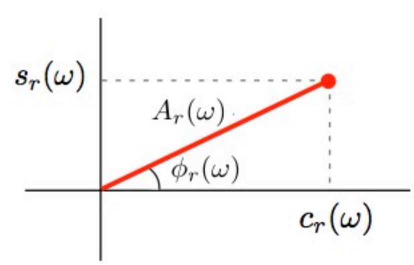

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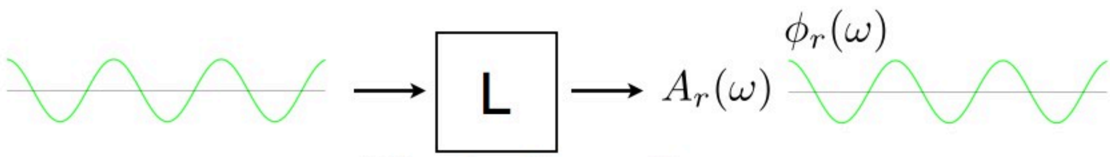
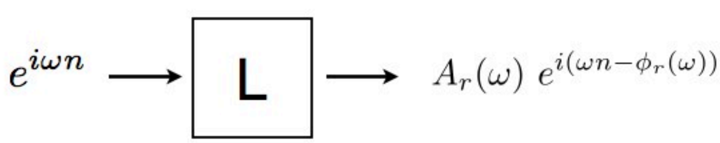


"Sine in, sine out"

Complex exponentials: "bundling" sine and cosine

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$



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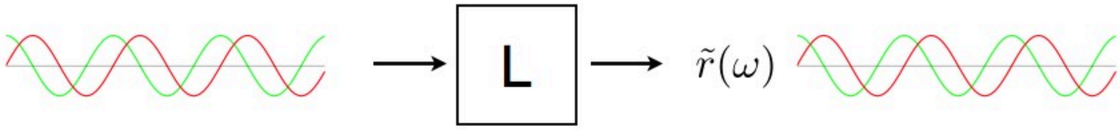

Discrete Fourier transform (DFT)

- Construct an orthogonal matrix of sin/cos pairs, at frequency multiples of $2\pi/N$ radians/sample, i.e., $2\pi k/N, \quad \text{for } k = 0, 1, 2, \ldots N/2$
- For $k = 0$ and $k = N/2$, only need the cosine part of the pair
- When we transform a vector using this matrix, think of output as paired coordinates
- Common to plot these pairs as amplitude/phase

Discrete Fourier transform (with complex numbers)

$$\tilde{r}_k = \sum_{n=0}^{N-1} r_n e^{-i\omega_k n}$$

$$r_n = \sum_{k=0}^{N-1} \tilde{r}_k \; e^{i\omega_k n} \qquad \text{(inverse)}$$

where $\quad \omega_k = \dfrac{2\pi k}{N}$

The Fourier family

signal domain

frequency domain

	continuous	discrete
continuous	**Fourier transform**	**discrete-time Fourier transform**
discrete	**Fourier series**	**discrete Fourier transform**

The "fast Fourier transform" (FFT) is a computationally efficient implementation of the DFT (runs in NlogN time, instead of N^2).

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you are here

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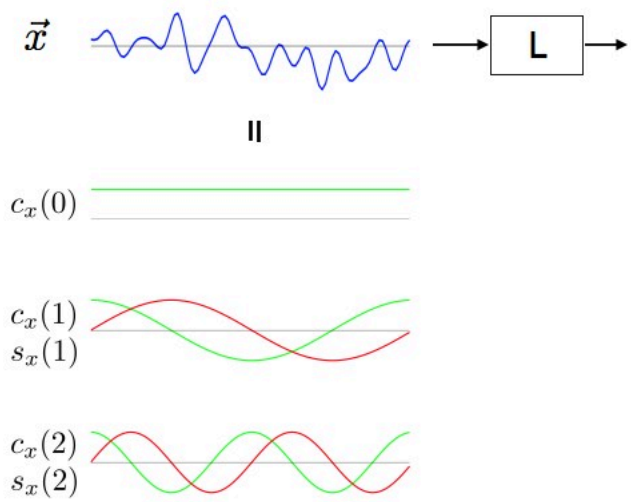
NOTE: Change in amplitude and phase come from Fourier transform of the impulse response!

Fourier & LSI

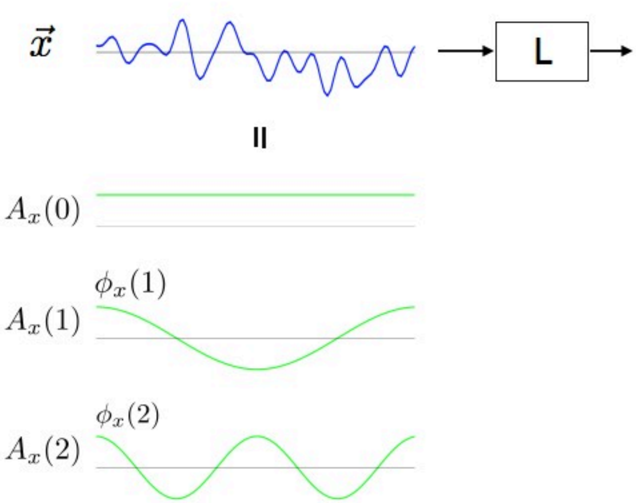
$\vec{x}$

Fourier & LSI



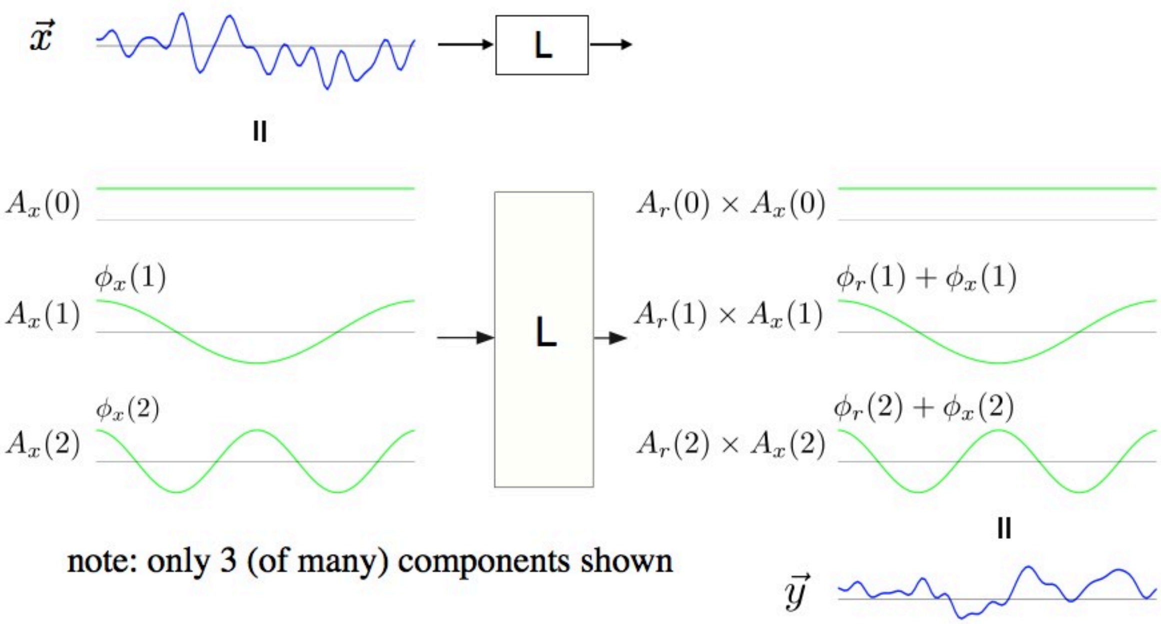
note: only 3 (of many) components shown

Fourier & LSI



note: only 3 (of many) components shown

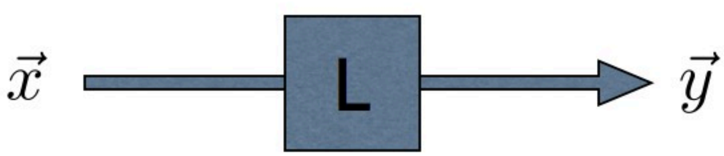
Fourier & LSI



note: only 3 (of many) components shown

LSI systems are characterized by their *frequency response*, specified by the Fourier Transform of their impulse response

The "convolution theorem"



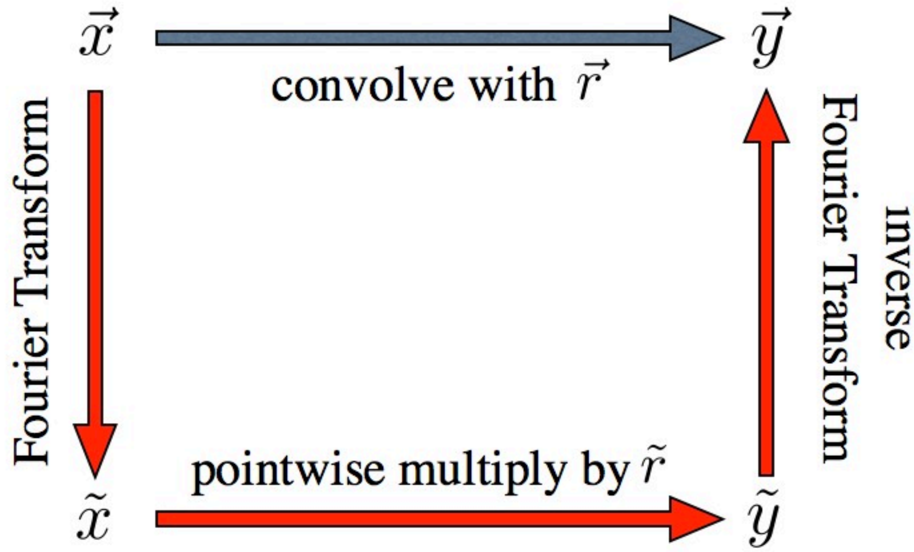
The “convolution theorem”

$$\vec{x} \longrightarrow \vec{y}$$

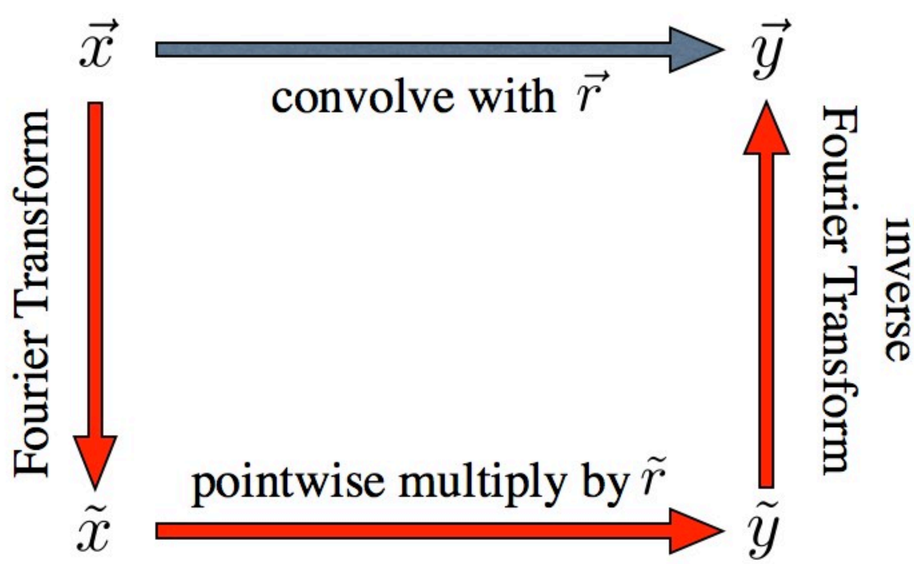
The “convolution theorem”

$$\vec{x} \xrightarrow[\text{convolve with } \vec{r}]{} \vec{y}$$

The "convolution theorem"



The "convolution theorem"

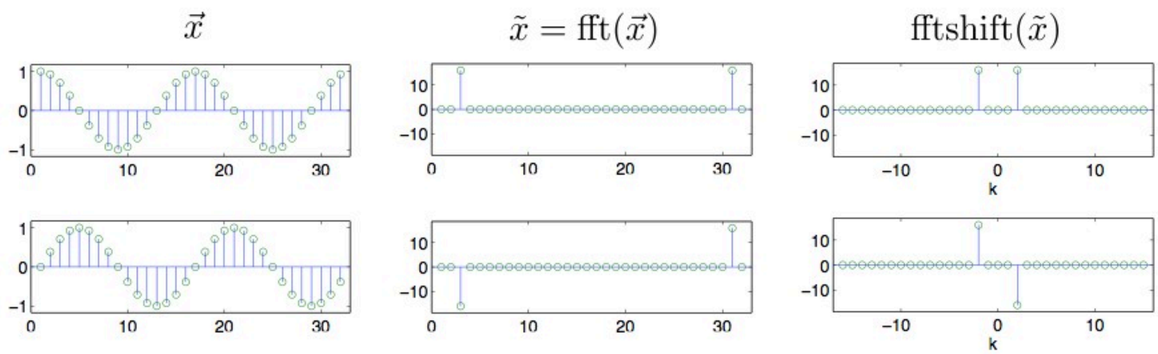


In matrix form: $L\vec{x} = FD(\vec{r})F^T\vec{x}$

$$e^{i\omega n} = \cos(\omega n) + i\sin(\omega n)$$

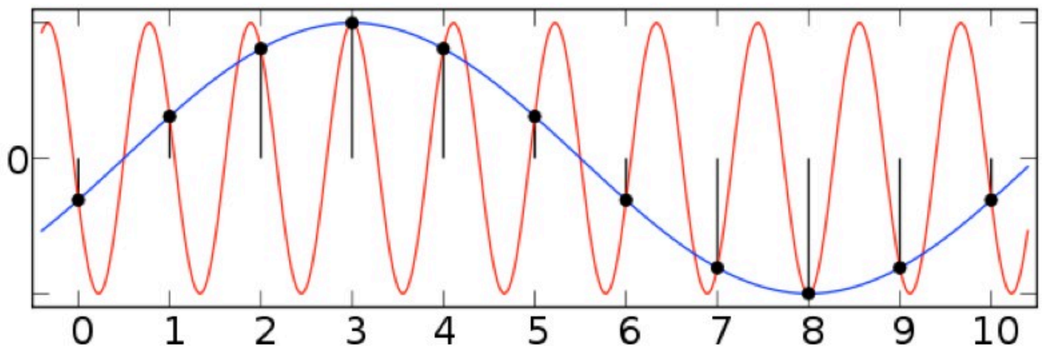
$$\cos(\omega n) = \frac{1}{2}(e^{i\omega n} + e^{-i\omega n}) \qquad \sin(\omega n) = \frac{-i}{2}(e^{i\omega n} - e^{-i\omega n})$$

DFT in Matlab, N=32 (note indexing and amplitudes):

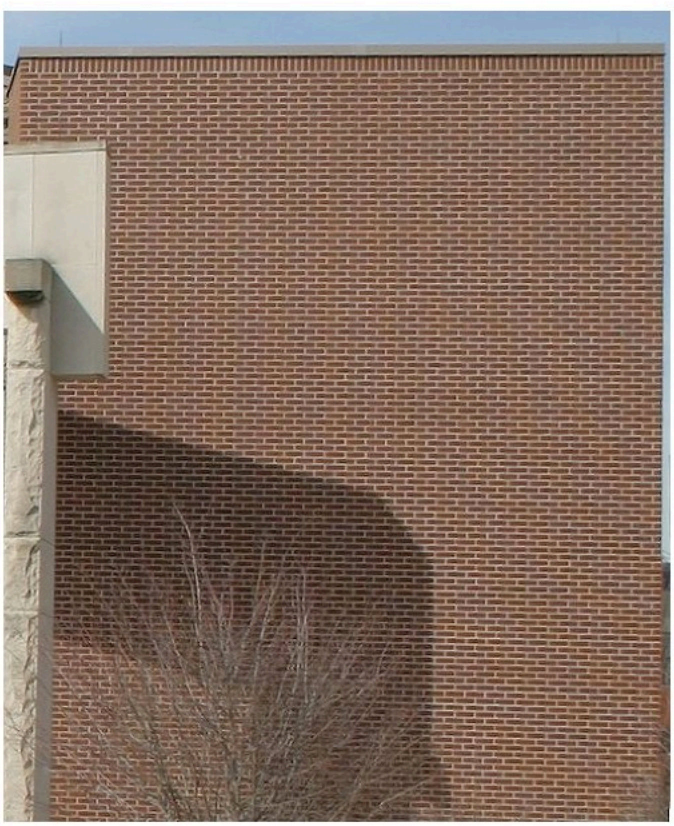


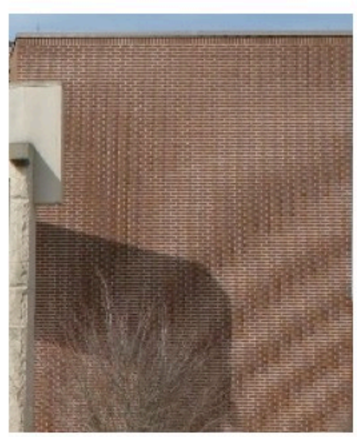
- Parseval
- translation
- dilation
- differentiation

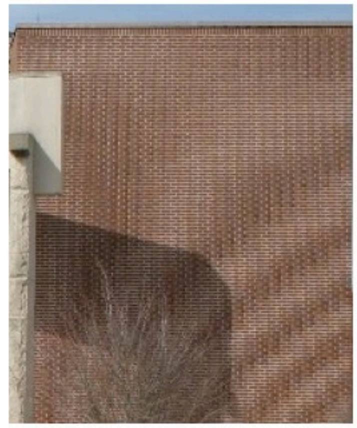
Sampling

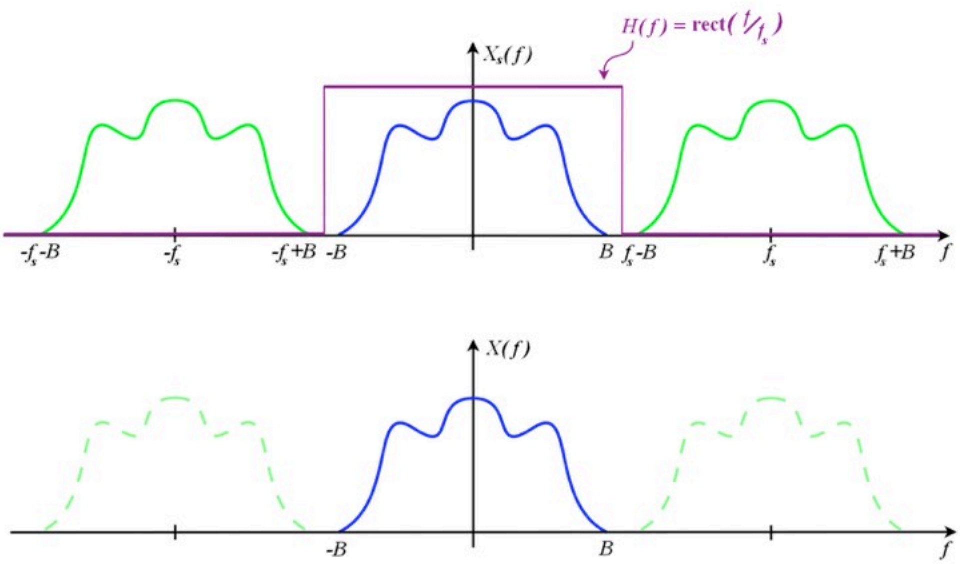


"Aliasing" (one frequency masquerades as another)



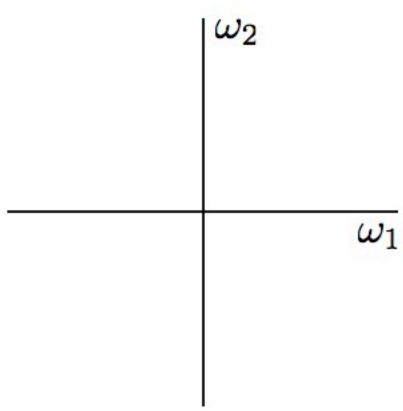






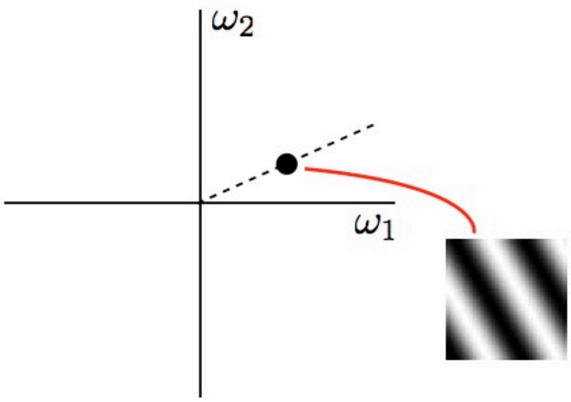
$H(f) = \mathrm{rect}(f/f_s)$
$X_s(f)$
$-f_s-B$
$-f_s$
$-f_s+B$
$-B$
B
f_s-B
f_s
f_s+B
f
$X(f)$
$-B$
B
f

Fourier, 2D



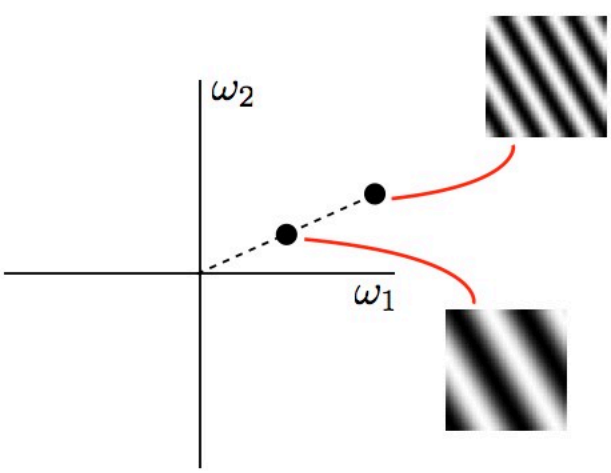
$$A \cos(\vec{\omega} \cdot \vec{x} - \phi)$$

Fourier, 2D



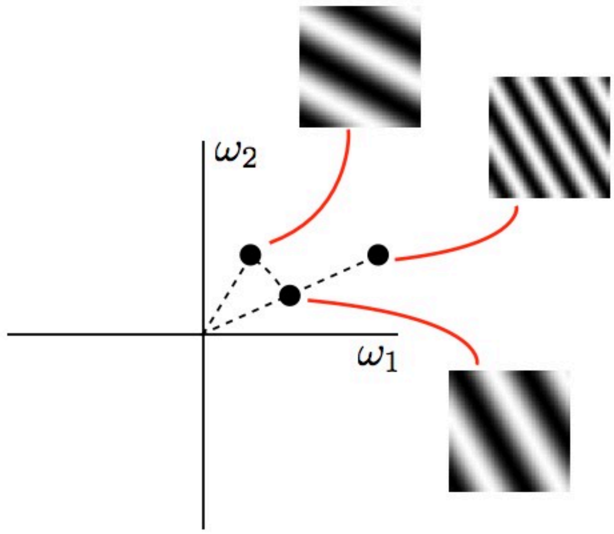
$$A\cos(\vec{\omega}\cdot\vec{x}-\phi)$$

Fourier, 2D



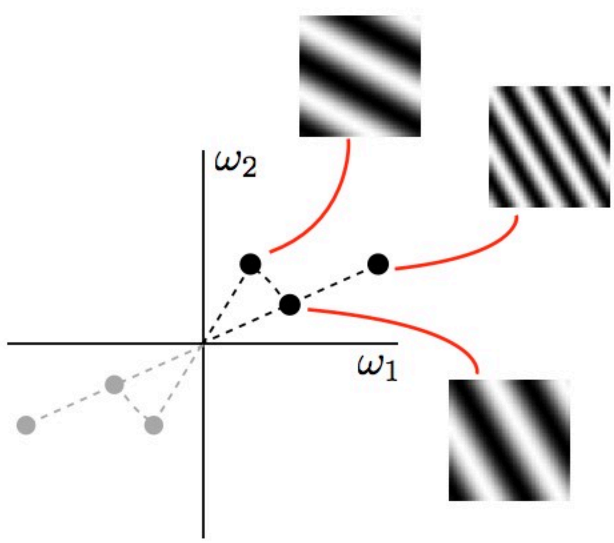
$$A\cos(\vec{\omega}\cdot\vec{x}-\phi)$$

Fourier, 2D



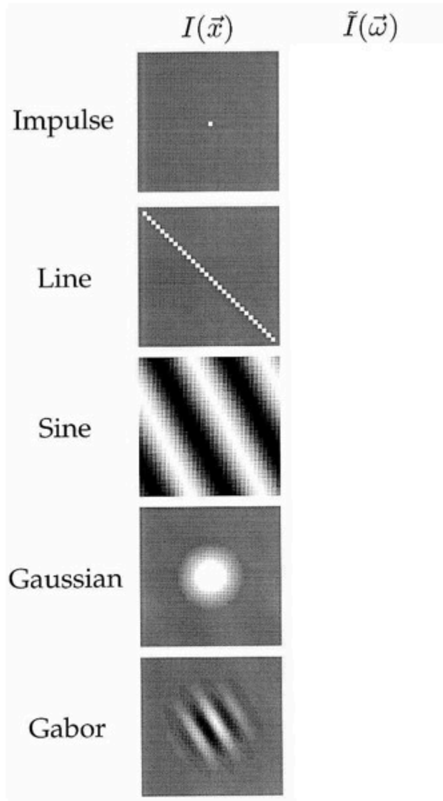
$A\cos(\vec{\omega}\cdot\vec{x}-\phi)$

Fourier, 2D

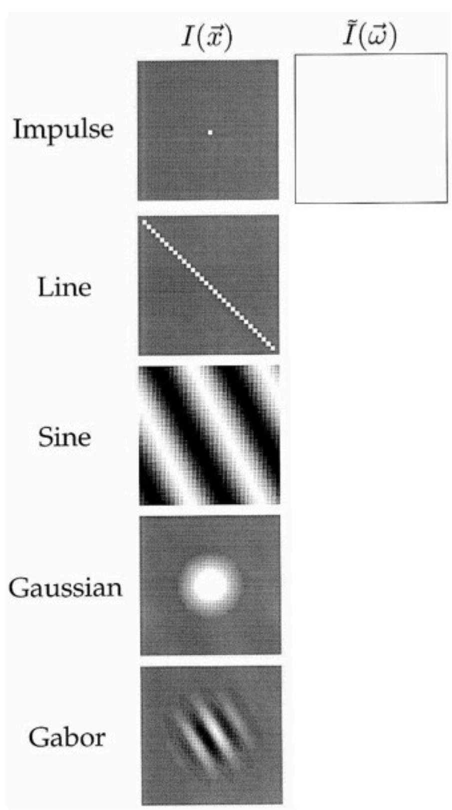


$A\cos(\vec{\omega}\cdot\vec{x}-\phi)$

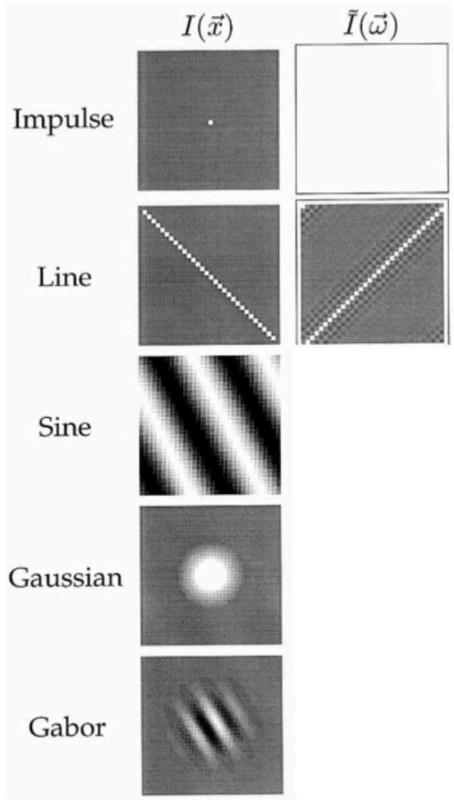
Fourier pairs



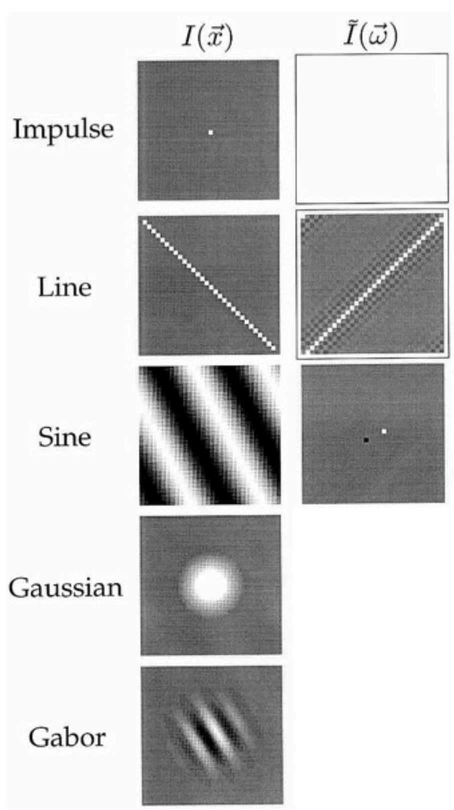
Fourier pairs



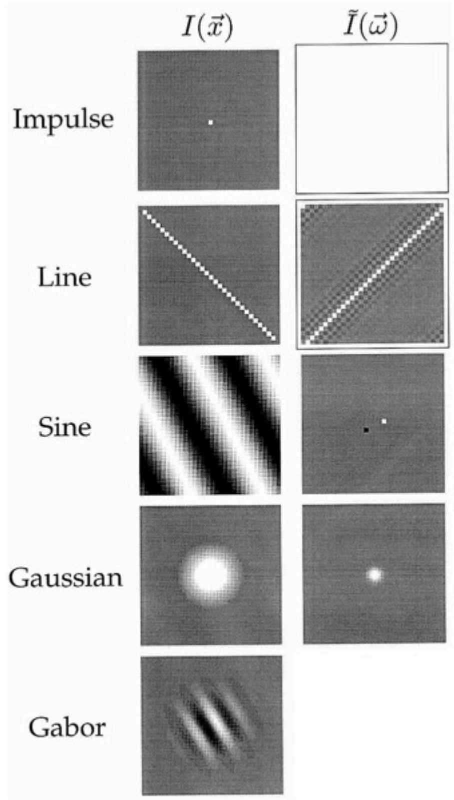
Fourier pairs



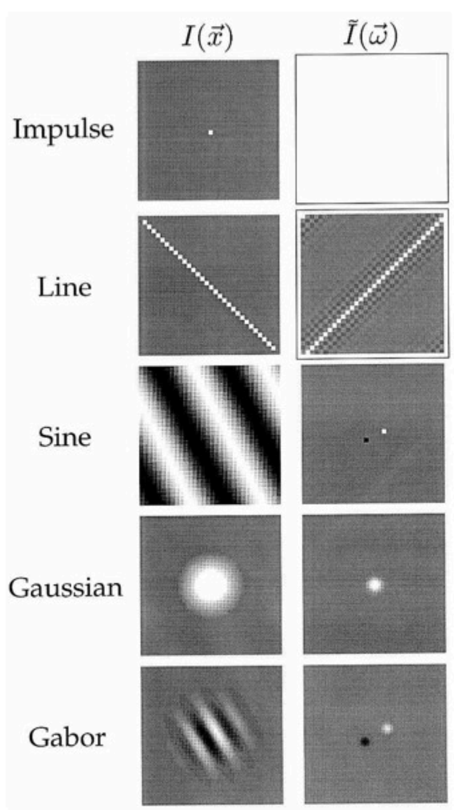
Fourier pairs



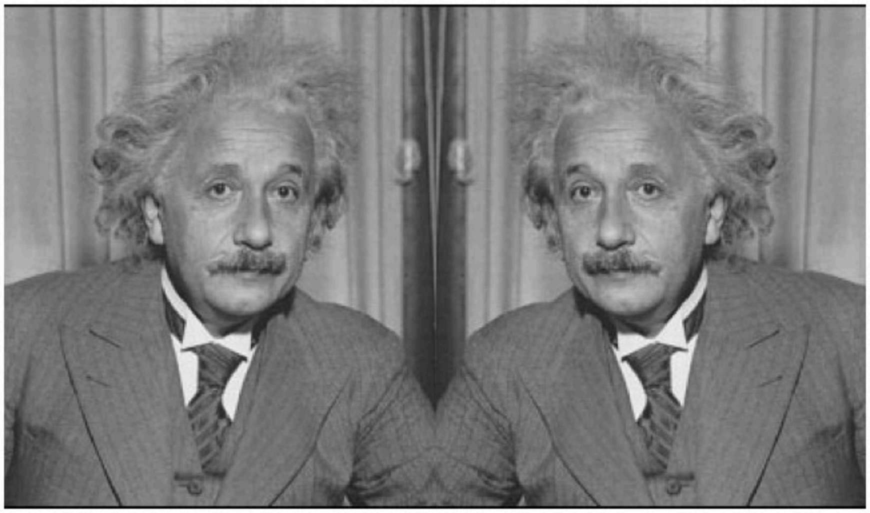
Fourier pairs



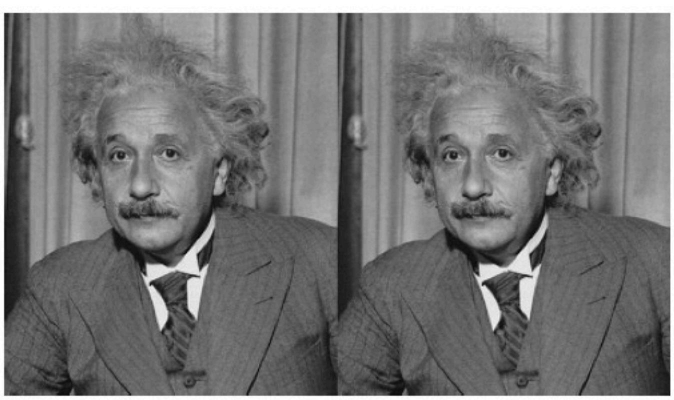
Fourier pairs

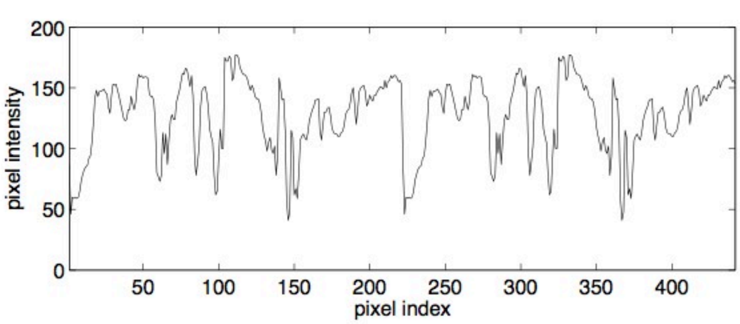


Boundaries?

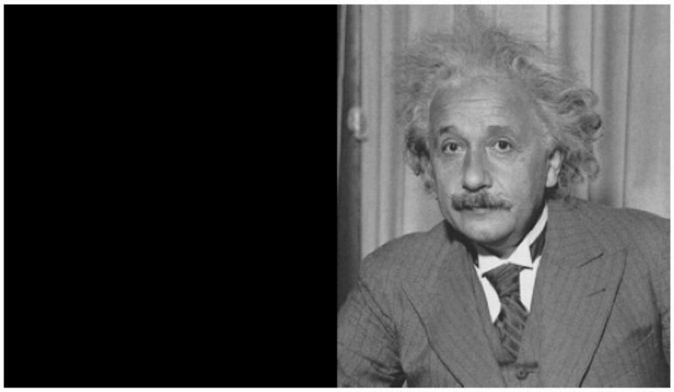


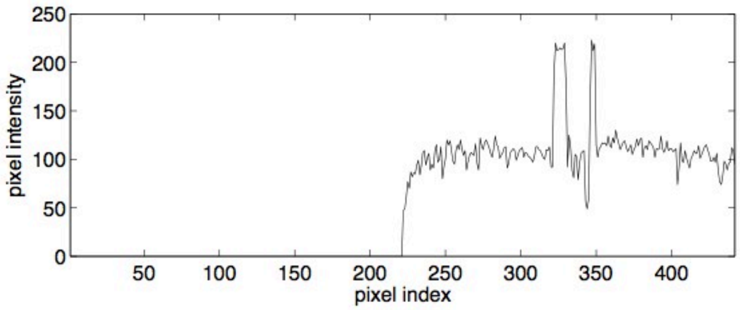
Boundary-handling: periodic

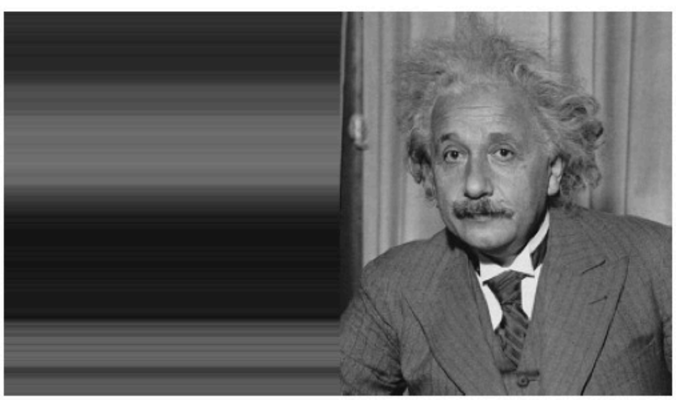


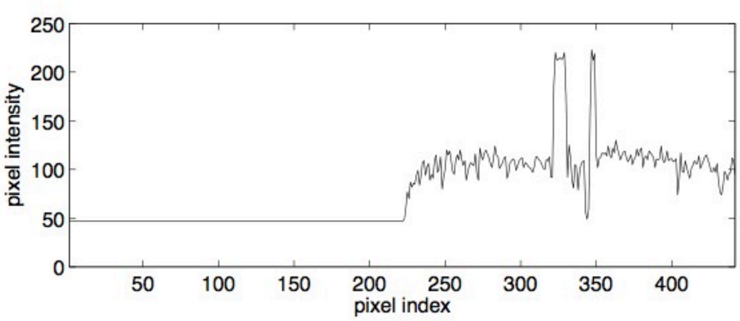


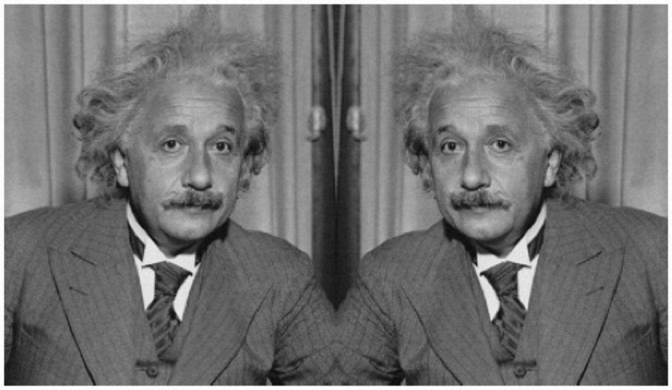
Boundary-handling?

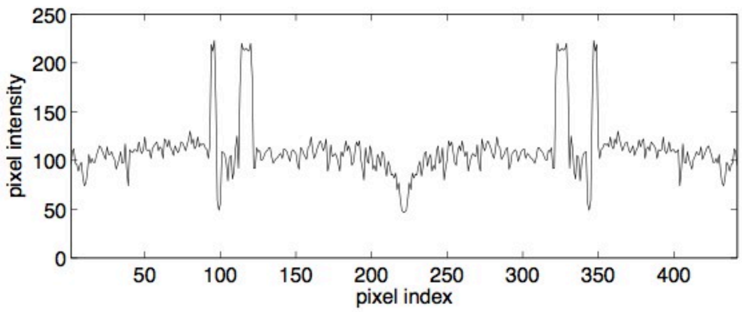








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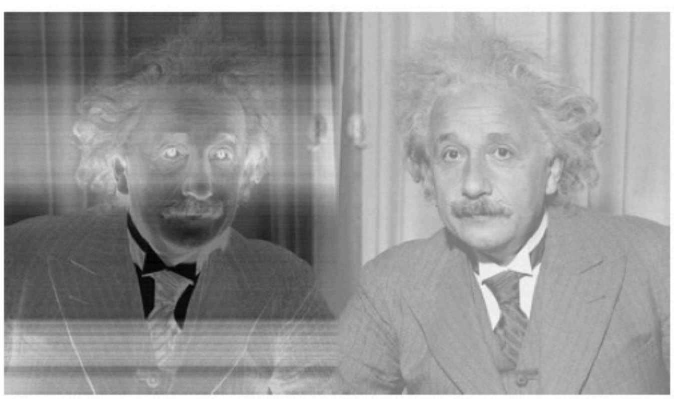
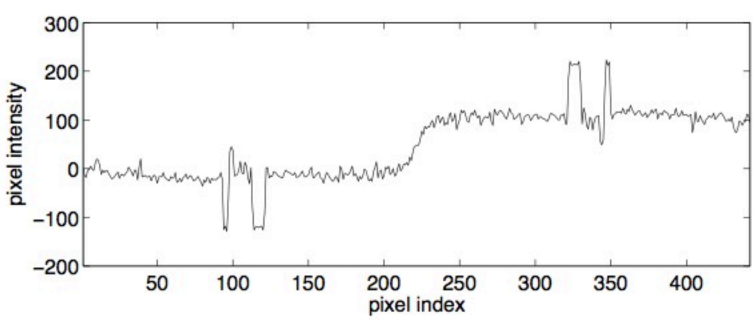
pixel index

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0
−100
−200
pixel intensity
50
100
150
200
250
300
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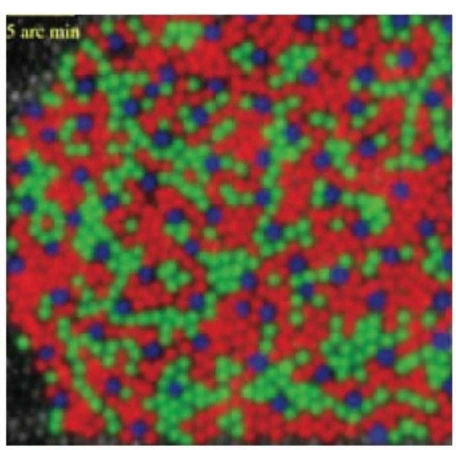
400
pixel index

2D sampling patterns

- square / rectangular
- quincunx / hexagonal

2D sampling patterns

- square / rectangular
- quincunx / hexagonal



Roorda and Williams (1999)