Relation Between 3D Invariants and 2D Invariants

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Abstract

Polynomial relations are established between the invariants of certain mixed sets of points and lines and the invariants of their projected images. The relations are obtained using the properties of a rational curve, in fact a twisted cubic, which is a covariant of the given set of points and lines.

1 Introduction

Let $\mathcal{C}$ be a geometrical configuration confined to a plane in space. For example, $\mathcal{C}$ could be a mixed set of points and lines or it could contain more general curves. Then certain properties of $\mathcal{C}$ are invariant or unchanged under collineations of the plane; in particular they are unchanged under projection to an image. These invariants are the basis of algorithms for deciding if a given image can be a projection of $\mathcal{C}$, or for reconstructing $\mathcal{C}$ from a sequence of images. The algorithms are efficient because the invariants relate the geometry of $\mathcal{C}$ directly to the image, without the necessity of considering camera position and orientation. The papers in [4], [5] describe many examples in which $\mathcal{C}$ contains points, lines or conics.

If $\mathcal{C}$ is not confined to a plane then the invariants of $\mathcal{C}$ are no longer related in a simple way to the invariants of an image. Let $\mathbf{v}_c$ be the invariants of $\mathcal{C}$ under collineations of projective space $\mathbb{P}^3$ and let $\mathbf{v}_i$ be the invariants of an image of $\mathcal{C}$ under collineations of the image plane $\mathbb{P}^2$. The simple equation $\mathbf{v}_c = \mathbf{v}_i$ holding in the planar case is replaced in many cases by a set of polynomial equations of the form $f(\mathbf{v}_c, \mathbf{v}_i) = 0$. If any of the equations are not satisfied then the image cannot be a projection of $\mathcal{C}$. If sufficiently many equations are obtained from a sequence of images, each image with a known vector of invariants $\mathbf{v}_i$, then the equations can be solved numerically to obtain $\mathbf{v}_c$. In many cases $\mathbf{v}_c$ is sufficient to determine $\mathcal{C}$ uniquely up to a collineation of $\mathbb{P}^3$.

In one of the simplest examples, $\mathcal{C}$ is a set of six points in general position in $\mathbb{P}^3$. In [6] a method is given for calculating vectors $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^6$ such that $\mathbf{a}$ depends only on $\mathbf{v}_c$, $\mathbf{b}$ depends only on $\mathbf{v}_i$, and such that the image is a projection of $\mathcal{C}$ if and only if $\mathbf{a}.\mathbf{b} = 0$.

A second example is described in [9], [10], [11]. It is assumed that the projection from space to the image is a generalised weak perspective transformation,

$$\mathbf{x} \mapsto H_1 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} H_2 \mathbf{x} \qquad (1)$$

where H_1 is an affine transformation of the image and H_2 is an affine transformation of space. The projection (1) is a good approximation to the camera projection when the depth variation of $\mathcal{C}$ is small compared to the distance from $\mathcal{C}$ to the camera and the image of $\mathcal{C}$ is near the centre of the field of view. Let $\mathcal{C}$ be set of points $\mathbf{x}_i$, $1 \leq i \leq n$, in $\mathbb{P}^3$ with coordinates chosen such that $(\mathbf{x}_i)_4 = 1$, $1 \leq i \leq n$, and let K be the $4 \times n$ matrix formed by taking the $\mathbf{x}_i$, $1 \leq i \leq n$ as columns. Let the image points be $\mathbf{q}_i$, $1 \leq i \leq n$, with coordinates in $\mathbb{P}^2$ chosen such that $(\mathbf{q}_i)_3 = 1$, $1 \leq i \leq n$, and let L be the $3 \times n$ matrix formed by taking the $\mathbf{q}_i$ as columns. Let $\mathcal{K}$, $\mathcal{L}$ be the right null spaces of K, L respectively. Then $\mathcal{K}$, $\mathcal{L}$ are affine invariants of $\mathbf{x}_i$, $\mathbf{q}_i$, and the $\mathbf{q}_i$ are a generalised weak projective transformation of the $\mathbf{x}_i$ if and only if $\mathcal{K} \subset \mathcal{L}$. The equations linking the invariants of the $\mathbf{x}_i$ and the invariants of the $\mathbf{q}_i$ are obtained from the condition $\mathcal{K} \subset \mathcal{L}$.

The method described below extends the first example in that it applies to mixed sets of points and lines. It also extends the second example in that the map from space to the image can be an arbitrary projection rather than a generalised weak perspective transformation.

The configurations $\mathcal{C}$ are required to be sets O_r of $r \leq 6$ points and $6 - r$ lines in general position in $\mathbb{P}^3$. The values $r = 0, 4$ are excluded for reasons which are explained in §3. A polynomial relation is established between the invariants of O_r under collineations of $\mathbb{P}^3$ and the invariants of a projected image I_r of O_r under collineations of $\mathbb{P}^2$. The basis of the polynomial relation is the observation that there is a unique twisted cubic which contains each point of O_r and which is bisecant to each line of O_r [3], [8].

2 Properties of Twisted Cubics

A brief summary of the relevant properties of twisted cubics is given. Further information can be found in [7], [8].

Let $\mathbf{t} \equiv (1, t, t^2, t^3)^\top$. A twisted cubic is any curve in $\mathbb{P}^3$ of the form $t \mapsto A\mathbf{t}$, where A is a non-singular 4×4 matrix. Let c_1, c_2 be the twisted cubics $t \mapsto A_1\mathbf{t}$, $t \mapsto A_2\mathbf{t}$. It follows that $A_2A_1^{-1}(c_1) = c_2$, thus any two twisted cubics are related by a collineation. The projective properties of any twisted cubic can be obtained from the standard twisted cubic defined by $t \mapsto \mathbf{t}$.

The parameterisation $t \mapsto A\mathbf{t} \in c$ establishes a birational map between the twisted cubic c and $\mathbb{P}$, thus c is a rational curve. The cross ratio, $\{\mathbf{x}_1, \mathbf{x}_2; \mathbf{x}_3, \mathbf{x}_4\}_c$, is well defined for any quadruple $\mathbf{x}_i$ of points on c. In more detail, let t_i be the parameter value corresponding to $\mathbf{x}_i$, $1 \leq i \leq 4$. Then the cross ratio is defined by

$$\{\mathbf{x}_1, \mathbf{x}_2; \mathbf{x}_3, \mathbf{x}_4\}_c = \left(\frac{t_1 - t_3}{t_1 - t_4}\right)\left(\frac{t_2 - t_4}{t_2 - t_3}\right)$$

It is invariant under any birational map of c, linear or otherwise. It is also invariant under reparameterisations $t \mapsto s(t)$ of c of the form

$$s = \frac{at + b}{ct + d} \quad (2)$$

where a, b, c, d are constants such that $ad - bc \neq 0$.

There is a three parameter family of collineations $\omega : \mathbb{P}^3 \to \mathbb{P}^3$ such that $\omega(c) = c$. These collineations correspond to the reparameterisations (2) in the following sense. For each such ω there exists a function $t \mapsto s(t)$ of the form (2) such that $\omega(A\mathbf{t}) = A\mathbf{s}$. Conversely, for each function (2) there exists a unique collineation $\omega : \mathbb{P}^3 \to \mathbb{P}^3$ such that $\omega(A\mathbf{t}) = A\mathbf{s}$.

Star generation is a striking and useful property of twisted cubics. Let $\mathbf{a}$, $\mathbf{b}$ be distinct points of $\mathbb{P}^3$, let $\mathrm{st}(\mathbf{a})$, $\mathrm{st}(\mathbf{b})$ be the stars of lines centred at $\mathbf{a}$, $\mathbf{b}$ and let $\omega : \mathrm{st}(\mathbf{a}) \to \mathrm{st}(\mathbf{b})$ be a general collineation. The locus of points $\mathbf{x}$ such that

$$\omega(\langle \mathbf{a}, \mathbf{x} \rangle) = \langle \mathbf{b}, \mathbf{x} \rangle \quad (3)$$

is a twisted cubic. Conversely, let $\mathbf{a}$, $\mathbf{b}$ be any two distinct points on a twisted cubic c. Then there is a unique collineation $\omega : \mathrm{st}(\mathbf{a}) \to \mathrm{st}(\mathbf{b})$ such that (3) holds for all $\mathbf{x}$ on c.

In algorithms based on star generation ω is often realised as a collineation of a plane in the following way. Let Π be a plane in $\mathbb{P}^3$ not containing $\mathbf{a}$ or $\mathbf{b}$. Each collineation $\omega : \mathrm{st}(\mathbf{a}) \to \mathrm{st}(\mathbf{b})$ corresponds to a collineation $\zeta : \Pi \to \Pi$ such that

$$\omega(\langle \mathbf{a}, \mathbf{x} \rangle) = \langle \mathbf{b}, \zeta(\langle \mathbf{a}, \mathbf{x} \rangle \cap \Pi) \rangle \quad (4)$$

A brief sketch is given of an algorithm for computing the parameterisation $t \mapsto A\mathbf{t}$ of c, given $\mathbf{a}$, $\mathbf{b}$, and ζ. The algorithm is readily implemented in Mathematica. Let coordinates be chosen such that $\mathbf{a} = (0, 0, 0, 1)^\top$, $\mathbf{b} = (0, 0, 1, 1)^\top$, and such that Π is the plane $x_4 = 0$. Let $\mathbf{p} = \langle \mathbf{a}, \mathbf{b} \rangle \cap \Pi$ and let $\mathbf{r}_t = (1, t, 0, 0)^\top$ be a variable point of Π. Let $\mathbf{u}$, $\mathbf{v}$, Ξ be defined by

$$\begin{aligned} \mathbf{u} &= \zeta^{-1}(\mathbf{v}) \\ \mathbf{v} &= \langle \mathbf{p}, \mathbf{r}_t \rangle \cap \langle \zeta(\mathbf{p}), \zeta(\mathbf{r}_t) \rangle \\ \Xi &= \langle \mathbf{a}, \mathbf{p}, \mathbf{r}_t \rangle \end{aligned}$$

Then the plane Ξ contains the line $\langle \mathbf{a}, \mathbf{u} \rangle$ because $\mathbf{u}$ is a point of $\langle \mathbf{p}, \mathbf{r}_t \rangle$. Similarly, Ξ contains $\langle \mathbf{b}, \mathbf{v} \rangle$. It follows that $\langle \mathbf{a}, \mathbf{u} \rangle$ and $\langle \mathbf{b}, \mathbf{v} \rangle$ intersect at some point $\mathbf{w}$. Then $\mathbf{w}$ is a point of c because $\zeta(\mathbf{u}) = \mathbf{v}$.

A short calculation yields

$$\mathbf{w} = (u_1v_1, u_2v_1, u_3v_1, u_3v_1 - u_1v_3)^\top$$

The components of $\mathbf{w}$ are quartic in t, but they have a linear greatest common factor which divides out leaving a vector $\mathbf{w}$ cubic in t. The matrix A is recovered from the equations $A\mathbf{t} = \mathbf{w}$.

3 Invariants of O_r

From now on the points of O_r are denoted by $\mathbf{x}_i$, $1 \leq i \leq r$ and the lines of O_r are denoted by l_i, $1 \leq i \leq 6 - r$.

There are many equivalent ways of describing the invariants of O_r under collineations of $\mathbb{P}^3$. In this application it is convenient to use a description based on the twisted cubic associated with O_r. It can be shown that if $r \neq 0, 4$, then there is a unique twisted cubic c that contains each point of O_r and that cuts each line of O_r twice. The fact that c is unique ensures that it is a covariant of O_r. In other words, if H is any collineation of $\mathbb{P}^3$, then $H(c)$ is the unique twisted cubic that contains each point of $H(O_r)$ and that cuts each line of $H(O_r)$ twice.

The cases $r = 0, 4$ are excluded. If $r = 0$, then there are in general exactly six twisted cubics bisecant to the lines of O_0. If $r = 4$, then the twisted cubic fails to exist.

Let $t \mapsto A\mathbf{t}$ be a parameterisation of c and let t_i, $1 \leq i \leq r$ be the values of t for which $\mathbf{x}_i = A\mathbf{t}_i$, and let u_i, v_i, $1 \leq i \leq 6 - r$ be the values of t at which c meets l_i. The lines of O_r are thus $l_i = \langle A\mathbf{u}_i, A\mathbf{v}_i \rangle$, $1 \leq i \leq 6 - r$. Each pair u_i, v_i is obtained from the roots of a quadratic equation, the coefficients of which are determined by A.

The invariants of O_r under collineations of $\mathbb{P}^3$ are polynomial functions of the cross ratios of the t_i, $1 \leq i \leq r$ and the u_i, v_i, $1 \leq i \leq 6 - r$. It is only necessary to choose the functions such that they are symmetric in each pair u_i, v_i. For example, the configuration O_3 has six fundamental invariants,

$$\begin{array}{ll} \{t_1, t_2; t_3, u_i\} + \{t_1, t_2; t_3, v_i\} & (i = 1, 2, 3) \\ \{t_1, t_2; t_3, u_i\}\{t_1, t_2; t_3, v_i\} & (i = 1, 2, 3) \end{array} \quad (5)$$

The invariants for the other configurations O_r, $r \neq 0, 4$ are constructed similarly.

The configurations O_r, $r \neq 0, 4$ are determined by the invariants up to a collineation of $\mathbb{P}^3$. In more detail, the twisted cubic c and three points on it can be chosen arbitrarily. If $r \geq 3$ then the three points can be chosen as $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$. Each pair u_i, v_i comprises the roots of a quadratic equation obtained from (5). If $r = 1, 2$ then $\mathbf{x}_1$, $A\mathbf{u}_1$, $A\mathbf{v}_1$ are chosen arbitrarily on c. The values of each pair u_i, v_i, $2 \leq i \leq 6 - r$ are obtained from the two invariants

$$\{t_1, u_1; v_1, u_i\} + \{t_1, v_1; u_1, u_i\} + \{t_1, u_1; v_1, v_i\} + \{t_1, v_1; u_1, v_i\} \quad (6)$$

and

$$\{t_1, u_1; v_1, u_i\}\{t_1, u_1; v_1, v_i\} + \{t_1, v_1; u_1, u_i\}\{t_1, v_1; u_1, v_i\} \quad (7)$$

The expressions (6), (7) are made symmetrical in u_1, v_1 so that a quadratic equation with roots u_i, v_i can be obtained without assigning particular values to u_1, v_1. It is only necessary to have expressions for u_1+v_1, u_1v_1 or equivalently to know the coefficients of a quadratic equation with roots u_1, v_1.

4 Construction of the Twisted Cubic

A method for constructing the twisted cubic associated with r is described briefly, for the cases $r \geq 2$, $r \neq 4$. Further details are given in [3], together with a discussion of the more difficult case $r = 1$.

Let $\mathbf{a} = \mathbf{x}_1$, $\mathbf{b} = \mathbf{x}_2$. Then there is a unique collineation $\omega : \mathrm{st}(\mathbf{a}) \rightarrow \mathrm{st}(\mathbf{b})$ such that

$$\begin{array}{rcll} \omega(\langle \mathbf{a}, \mathbf{x}_i \rangle) & = & \langle \mathbf{b}, \mathbf{x}_i \rangle & (3 \leq i \leq r) \\ \omega(\langle \mathbf{a}, l_i \rangle) & = & \langle \mathbf{b}, l_i \rangle & (1 \leq i \leq 6 - r) \quad (8) \end{array}$$

For example, in the case $r = 6$ let Π be a plane not containing $\mathbf{a}$, $\mathbf{b}$. Let $\mathbf{w}_i$, $\mathbf{z}_i$ be defined by

$$\begin{array}{rcll} \mathbf{w}_i & = & \langle \mathbf{a}, \mathbf{x}_i \rangle \cap \Pi & (3 \leq i \leq 6) \\ \mathbf{z}_i & = & \langle \mathbf{b}, \mathbf{x}_i \rangle \cap \Pi & (3 \leq i \leq 6) \end{array}$$

Then there is a unique collineation $\zeta : \Pi \rightarrow \Pi$ such that $\zeta(\mathbf{w}_i) = \mathbf{z}_i$, $3 \leq i \leq 6$. The collineation ω is given by (4). The twisted cubic c through the six points $\mathbf{x}_i$ is constructed from $\mathbf{a}$, $\mathbf{b}$, ζ using the algorithm sketched at the end of §2.

Let $\mathbf{p}$, $\mathbf{r}_t$ be as defined in §2. The parameter value t_i corresponding to $\mathbf{x}_i$ on c is determined by the condition that $\mathbf{p}$, $\mathbf{r}_{t_i}$ and $\mathbf{w}_i$ are collinear. The parameter values u_i, v_i corresponding to the two points $c \cap l_i$ are obtained as follows. Let $\mathbf{m}$, $\mathbf{n}$ be distinct known points on l_i, and let $\mathbf{e}_\lambda$, $\mathbf{f}_\lambda$ be points defined by

$$\begin{array}{rcl} \mathbf{e}_\lambda & = & \mathbf{m} + \lambda \mathbf{n} \\ \mathbf{f}_\lambda & = & \zeta(\langle \mathbf{a}, \mathbf{e}_\lambda \rangle \cap \Pi) \end{array}$$

where λ is a parameter. Then $\mathbf{e}_\lambda$ is a point of $c \cap l_i$ if and only if $\mathbf{a}$, $\mathbf{b}$, $\mathbf{e}_\lambda$, $\mathbf{f}_\lambda$ are coplanar. This condition yields a quadratic constraint on λ, the roots of which yield the two points of $c \cap l_i$. It remains to relate λ to the parameter t for c. The function $t \mapsto \lambda(t)$ is chosen such that $\mathbf{m} + \lambda \mathbf{n} = \langle \mathbf{a}, \mathbf{b}, A\mathbf{t} \rangle \cap l_i$. The points $\mathbf{a}$, $\mathbf{b}$, $A\mathbf{t}$, $\mathbf{m} + \lambda \mathbf{n}$ are coplanar thus

$$\det(\mathbf{a}, \mathbf{b}, A\mathbf{t}, \mathbf{m} + \lambda \mathbf{n}) = 0 \quad (9)$$

Equation (9) is cubic in $\mathbf{t}$. It has two factors independent of λ, and yielding values of t for which $A\mathbf{t} = \mathbf{a}$ or $A\mathbf{t} = \mathbf{b}$. The third factor is bilinear in t and λ.

The fact that no twisted cubic exists for $r = 4$ is apparent from (8). Let $h_i = \langle \mathbf{a}, l_i \rangle \cap \Pi$, $k_i = \langle \mathbf{b}, l_i \rangle \cap \Pi$, $i = 1, 2$, $h = \langle \mathbf{w}_1, \mathbf{w}_2 \rangle$, $k = \langle \mathbf{z}_1, \mathbf{z}_2 \rangle$. Then it is necessary to find $\zeta : \Pi \rightarrow \Pi$ such that $\zeta(\mathbf{w}_i) = \mathbf{z}_i$ and $\zeta(h_i) = k_i$, $i = 1, 2$. In general ζ does not exist because the cross ratio of the four collinear points $\mathbf{w}_1$, $\mathbf{w}_2$, $h \cap h_1$, $h \cap h_2$ is different from the cross ratio of the four collinear points $\mathbf{z}_1$, $\mathbf{z}_2$, $k \cap k_1$, $k \cap k_2$.

The algorithm for constructing c from O_r can be made exact in the following sense. If the points of O_r have rational coordinates and if the lines of O_r are defined by pairs of points also with rational coordinates then c has a parameterisation $t \mapsto A\mathbf{t}$ in which A has rational entries. These entries can be computed exactly. It is only necessary to factor polynomials and to solve linear equations.

5 Relation of Object Invariants to Image Invariants

The invariants of sets of points and lines with coordinates $\mathbf{q}_i$, $\mathbf{k}_i$ in $\mathbb{P}^2$ are constructed from the following fundamental expressions [1]

$$\begin{array}{rcl} I_{ijk} & = & \det(\mathbf{q}_i, \mathbf{q}_j, \mathbf{q}_k) \\ J_{ij} & = & \mathbf{k}_i . \mathbf{q}_j \\ K_{ijk} & = & \det(\mathbf{k}_i, \mathbf{k}_j, \mathbf{k}_k) \end{array} \quad (10)$$

The link between the invariants of O_r and the invariants of the image I_r depends on a result of Carlsson [2]. Let coordinates be chosen for $\mathbb{P}^3$ and for the image $\mathbb{P}^2$, and let P be the 3×4 matrix defining the projection from space to the image. Let $\mathbf{o}$ be the optical centre of P, and let D be the 4×4 matrix obtained from P by adding $\mathbf{o}$ as a fourth row. Carlsson shows that

$$\det(D(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{o})) = \|\mathbf{o}\|^2 \det(P\mathbf{x}_1, P\mathbf{x}_2, P\mathbf{x}_3) \quad (11)$$

Let coordinates be chosen in $\mathbb{P}^3$ such that the twisted cubic associated with O_r is $t \mapsto \mathbf{t}$. Let $P\mathbf{t}_i = \mathbf{q}_i$, $k_i = \langle P\mathbf{u}_i, P\mathbf{v}_i \rangle$, with the consequence $\mathbf{k}_i = P\mathbf{u}_i \times P\mathbf{v}_i$. It follows from (10) and (11) that

$$\begin{array}{rcl} I_{ijk} & = & \det(P\mathbf{t}_i, P\mathbf{t}_j, P\mathbf{t}_k) \\ & = & \|\mathbf{o}\|^{-2} \det(D(\mathbf{t}_i, \mathbf{t}_j, \mathbf{t}_k, \mathbf{o})) \\ J_{ij} & = & \det(P\mathbf{u}_i, P\mathbf{v}_i, P\mathbf{t}_j) \\ & = & \|\mathbf{o}\|^{-2} \det(D(\mathbf{u}_i, \mathbf{v}_i, \mathbf{t}_j, \mathbf{o})) \\ K_{ijk} & = & (\mathbf{k}_i \times \mathbf{k}_j) . \mathbf{k}_k \\ & = & \det(P\mathbf{u}_i, P\mathbf{v}_i, P\mathbf{v}_j) \det(P\mathbf{u}_k, P\mathbf{v}_k, P\mathbf{u}_j) \\ & & - \det(P\mathbf{u}_i, P\mathbf{v}_i, P\mathbf{u}_j) \det(P\mathbf{u}_k, P\mathbf{v}_k, P\mathbf{v}_j) \end{array}$$

$$= \|\mathbf{o}\|^{-4} \det(D(\mathbf{u}_i, \mathbf{v}_i, \mathbf{v}_j, \mathbf{o})) \det(D(\mathbf{u}_k, \mathbf{v}_k, \mathbf{u}_j, \mathbf{o}))$$

$$- \|\mathbf{o}\|^{-4} \det(D(\mathbf{u}_i, \mathbf{v}_i, \mathbf{u}_j, \mathbf{o})) \det(D(\mathbf{u}_k, \mathbf{v}_k, \mathbf{v}_j, \mathbf{o}))$$

$$(12)$$

The image invariants are ratios of the I_{ijk}, J_{ij}, K_{ijk} constructed such that the scale factors involved in the choice of coordinates in $\mathbb{P}^2$ all cancel. For example a set of five coplanar points has an invariant

$$v_1 = \frac{I_{123}I_{145}}{I_{124}I_{135}}$$

thus

$$v_1 = \frac{\det(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3, \mathbf{o})\det(\mathbf{t}_1, \mathbf{t}_4, \mathbf{t}_5, \mathbf{o})}{\det(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_4, \mathbf{o})\det(\mathbf{t}_1, \mathbf{t}_3, \mathbf{t}_5, \mathbf{o})}$$

which yields a polynomial equation

$$f(t_1, t_2, t_3, t_4, t_5, \mathbf{o}, v_1) = 0$$

quadratic in $\mathbf{o}$. If t_1, t_2, t_3 are assigned the standard values 0, 1, ∞, then the remaining t_i and the pairs $u_i + v_i$, $u_i v_i$ are invariants of O_r.

A set of two points and two lines has the single invariant

$$v_2 = \frac{J_{11}J_{22}}{J_{12}J_{21}}$$

It follows from (11) and (12) that

$$\begin{aligned} v_2 &= \frac{\det(P\mathbf{u}_1, P\mathbf{v}_1, P\mathbf{t}_1)\det(P\mathbf{u}_2, P\mathbf{v}_2, P\mathbf{t}_2)}{\det(P\mathbf{u}_1, P\mathbf{v}_1, P\mathbf{t}_2)\det(P\mathbf{u}_2, P\mathbf{v}_2, P\mathbf{t}_1)} \\ &= \frac{\det(D(\mathbf{u}_1, \mathbf{v}_1, \mathbf{t}_1, \mathbf{o}))\det(D(\mathbf{u}_2, \mathbf{v}_2, \mathbf{t}_2, \mathbf{o}))}{\det(D(\mathbf{u}_1, \mathbf{v}_1, \mathbf{t}_2, \mathbf{o}))\det(D(\mathbf{u}_2, \mathbf{v}_2, \mathbf{t}_1, \mathbf{o}))} \\ &= \frac{\det(\mathbf{u}_1, \mathbf{v}_1, \mathbf{t}_1, \mathbf{o})\det(\mathbf{u}_2, \mathbf{v}_2, \mathbf{t}_2, \mathbf{o})}{\det(\mathbf{u}_1, \mathbf{v}_1, \mathbf{t}_2, \mathbf{o})\det(\mathbf{u}_2, \mathbf{v}_2, \mathbf{t}_1, \mathbf{o})} \qquad (13) \end{aligned}$$

Equation (13) yields

$$g(t_1, t_2, u_1, v_1, u_2, v_2, \mathbf{o}) = 0$$

where g is a polynomial, quadratic in $\mathbf{o}$.

The remaining types of invariant are

$$\frac{K_{123}J_{41}}{K_{124}J_{31}} \qquad \frac{K_{123}K_{145}}{K_{124}K_{135}}$$

The first of these yields a cubic equation in $\mathbf{o}$ and the second yields a quartic equation in $\mathbf{o}$.

The image I_r has a total of four independent invariants, yielding four polynomial constraints on $\mathbf{o}$. The vector $\mathbf{o}$ can be eliminated to yield a single polynomial equation in the invariants of O_r and the invariants of I_r.

6 Conclusion

A polynomial constraint has been obtained between the invariants of certain configurations O_r of r points and $6 - r$ lines ($r \neq 0, 4$) in $\mathbb{P}^3$ and the invariants of an image of O_r. The map from space to the image is assumed to be a projection, but it is not otherwise constrained. The basis of the polynomial constraint is the observation that there is a unique twisted cubic which contains each point of O_r and which cuts each line of O_r twice, $r \neq 0, 4$.

There are many questions which arise from this work, for example:

i) Are the polynomial constraints a basis for practical algorithms either to reconstruct O_r from multiple views or to recognise O_r from a small number of views?

ii) Can the methods be extended to include cases in which the camera calibration (internal orientation) is constant but unknown?

iii) Is the use of a twisted cubic to define invariants a pointer to more general methods for reconstruction from image sequences, possibly involving other rational curves, or is it a 'trick' applicable only in this special case?

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