Logarithms

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A logarithm is an exponent. Exponents are familiar to everyone who has had high school mathematics. Briefly, they are numbers usually written as a superscript whose meaning can be specified as follows (*n* and *m* are integers):

Symbol Meaning	Example
$a^n = a \times a \times \cdots \times a \quad (n \text{ times})$	$7^4 = 7 \times 7 \times 7 \times 7 = 2401$
$a^{-n} = 1/a^n$	$5^{-3} = 1/(5 \times 5 \times 5) = 1/125$
$a^{1/m} = \sqrt[m]{a}$	$27^{1/3} = \sqrt[3]{27} = 3, \text{ since } 3^3 = 3 \times 3 \times 3 = 27$
$a^{n/m} = \sqrt[m]{a^n}$	$4^{3/6} = \sqrt[6]{4^3} = \sqrt[6]{64} = 2, \text{ since } 2^6 = 64$

If *b* is any positive number, not equal to 1, and

$$b^x = a,$$

then the exponent *x* is called the logarithm of *a* to the base *b*. Given any positive real number *a* and base *b* (where $b \neq 1$), this equation will have a solution *x*, so every positive real number has a logarithm to any base. We will be concerned only with the logarithms of positive real numbers. The logarithms themselves may be positive or negative as we will see. We write

$$\log_b a = x.$$

All "$\log_b a$" means is the exponent that *b* would have to be raised to in order to equal *a*.

The base 10 is the most commonly used in vision research, and logarithms to the base 10 are called *common logarithms*. In writing common logarithms, sometimes the base is omitted from the notation. Thus, the expression $\log a$ means $\log_{10} a$.

There are three basic theorems on logarithms each of which is simply a translation (from the language of exponents into the language of logarithms) of a well-known theorem for exponents.

Theorems

For Exponents	For Logarithms
$b^p b^q = b^{p+q}$	$\log xy = \log x + \log y$
$b^p / b^q = b^{p-q}$	$\log x / y = \log x - \log y$
$\left(b^p\right)^q = b^{pq}$	$\log x^y = y \log x$

These theorems show that, by the use of logarithms, computations can be greatly simplified. Multiplication corresponds to the addition of logarithms, division, to subtraction; and exponentiation, to multiplication.

For certain carefully selected numbers $\log_{10}$ is very easy to compute:

$$\log_{10} 10 = 1, \text{ since } 10^1 = 10$$

$$\log_{10} 100 = 2, \text{ since } 10^2 = 100$$

$$\log_{10} .01 = -2, \text{ since } 10^{-2} = .01$$

$$\log_{10} 10^n = n, \text{ since } 10^n = 10^n$$

For most numbers, however, it is not at all obvious what their logarithms to the base 10 are, nor is there any simple way to compute them (other than, of course, by use of a calculator!). However, any number can be expressed as the product of a number between 1 and 10 times 10 raised to an integral power. For example:

$$1476 = 1.476 \times 10^3.$$

Therefore:

$$\log 1476 = \log\left(1.476 \times 10^3\right) = \log 1.476 + \log 10^3 = 3 + \log 1.476.$$

So, if we know the common logarithms of numbers between 1 and 10, it will be easy to determine the logarithm of any number. The logarithms of numbers between 1 and 10 are called *mantissas*. They are what are given (approximately) in a table of logarithms. They are all *positive decimals*, since 10 must be raised to some positive power less than 1 to give a number between 1 and 10. In a table of mantissas, we find:

$$\log_{10} 1.476 = .169,$$

therefore,

$$\log_{10} 1476 = \log 10^3 + \log 1.476 = 3 + .169 = 3.169.$$

When the logarithm of any number is written in this way, the integer is called its *characteristic* and the decimal is called its *mantissa*. Note that logarithms of numbers which differ only in the position of the decimal point all have the same mantissa.

$$\log 14.76 = \log 10 \times 1.476 = 1.169$$

$$\log 147.6 = \log 10^2 \times 1.476 = 2.169.$$

When the number is less than 1, the characteristic will be negative.

$$\log .001476 = \log 10^{-3} \times 1.476 = -3 + .169 = -2.831.$$

Antilogarithms

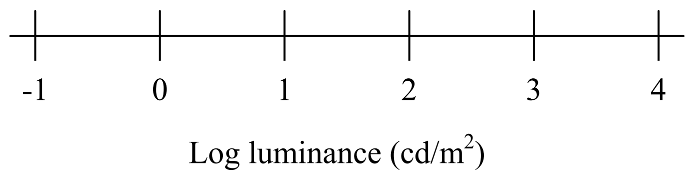
It often happens that we know a logarithm x, and want to determine the number a which has x for its logarithm. If $b^x = a$, then a is called the antilogarithm of x to the base b. Finding an antilogarithm is, in a sense, the inverse of finding a logarithm.

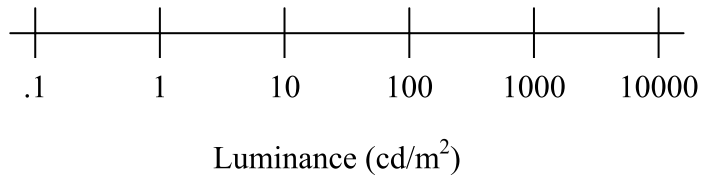
Concept of a log unit

The phrase “log unit” is frequently used in perception research. An example is the statement that one threshold is two log units greater than another. What this means is that the difference between the logarithms of the threshold values is 2. This is equivalent to saying that one threshold is 100 times the other. Values that differ by 1 log unit have a ratio of 10 to 1; those that differ by 3 log units have a ratio of 1000 to 1, etc. Note that it makes no sense to say that a value is “2 log units.” The number of log units must always be relative to some other value

Why are logarithms useful?

The intensity of light can be expressed in any of several units, the most common of which is cd/m^2. When we look at research reports, however, we find that it is log intensity (e.g., log cd/m^2) rather than intensity which is graphed. Alternatively, the numbers will indicate cd/m^2 but they will be plotted on a logarithmic scale. That is, they will be plotted so that values that differ by the same number of log units will be equally spaced.





In auditory research, intensities are almost always expressed in decibels relative to a reference intensity. A decibel is .1 of a log unit of power or energy.

There are at least three important reasons why logarithms, log scales, and logarithmic units are used in perception research.

The first is that the response of perceptual systems is roughly a logarithmic function of stimulus intensity. This means that at low intensities the response changes more than at high intensities for a constant intensity change. To determine these response changes it is necessary to sample many intensities in the low intensity range. Fewer are needed in the high intensity range. A common way to achieve this is to use intensities that are equally spaced on a log scale. To illustrate the low intensity changes graphically, low intensities must be spread out on the graph. This is done by graphing on a logarithmic scale.

The second reason for using logarithms in sensory work is that variables that influence functional relations often have multiplicative effects. If functions are plotted on ordinary coordinates, it is usually very difficult to tell whether they differ by a constant or not. Logarithmic coordinates, however, turn multiplication into addition and in log coordinates functions which differ by a constant multiple will be parallel. Parallelness is easy to judge visually.

The third reason for using logarithms is that devices commonly used to control intensity act in such a way as to decrease or increase intensity in steps that differ by multiplicative factors. Filters used to control light intensity reduce intensity at any wavelength to a constant proportion of its former value. The proportion may be the same at all wavelengths (a neutral filter) or vary with wavelength (a color filter), but the proportion is independent of the intensity of the light. A neutral filter can be characterized by a single number, its transmission T. This is simply the proportion of the light reaching it which passes through. But it is more convenient to characterize a filter in terms of its density D, where:

$$D = \log(1/T) = \log 1 - \log T = -\log T.$$

The luminance output of a filter L_o can be expressed in terms of luminance input L_i and filter density as follows:

$$\log L_o = \log L_i T = \log L_i + \log T = \log L_i - D.$$

So, if $\log L_i$ is known, one simply subtracts D to obtain $\log L_o$.

The intensity of auditory stimuli is usually controlled by attenuators which vary in decibel steps.

Facts about logarithms

The following facts about logarithms are useful to know:

If x=y, then $\log x = \log y$
If $x = \log y$, then $y = 10^x$
$\log 10^x = x$
$\log x^y = y \log x$
$\log xy = \log x + \log y$
$\log \frac{x}{y} = \log x - \log y$
$\log .1 = -1$
$\log 1 = 0$
$\log 10 = 1$
Only take logs for x,y>0