Linear, Shift-Invariant Systems

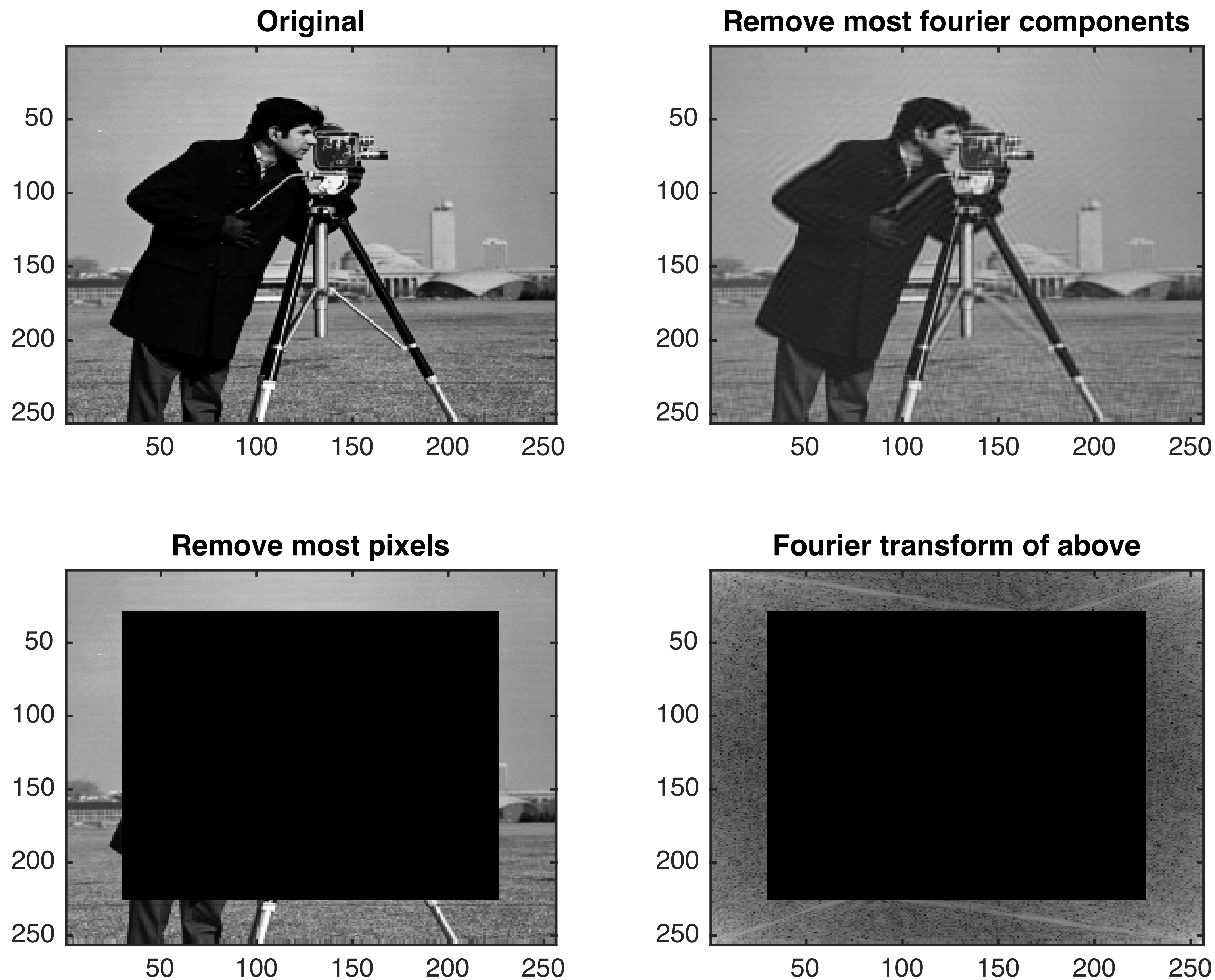
- Linearity: scalar rule and additivity
- Applied to impulse, sums of impulses
- Applied to sine waves, sums of sine waves

Summary: Linear Systems Theory

- Signals can be represented as sums of sine waves
- Linear, shift-invariant systems operate “independently” on each sine wave, and merely scale and shift them.
- A simplified model of neurons in the visual system, the linear receptive field, results in a neural image that is linear and shift-invariant.
- Psychophysical models of the visual system might be built of such mechanisms.
- It is therefore important to understand visual stimuli in terms of their spatial frequency content.
- The same tools can be applied to other modalities (e.g., audition) and other signals (EEG, MRI, MEG, etc.).

From lecture 1

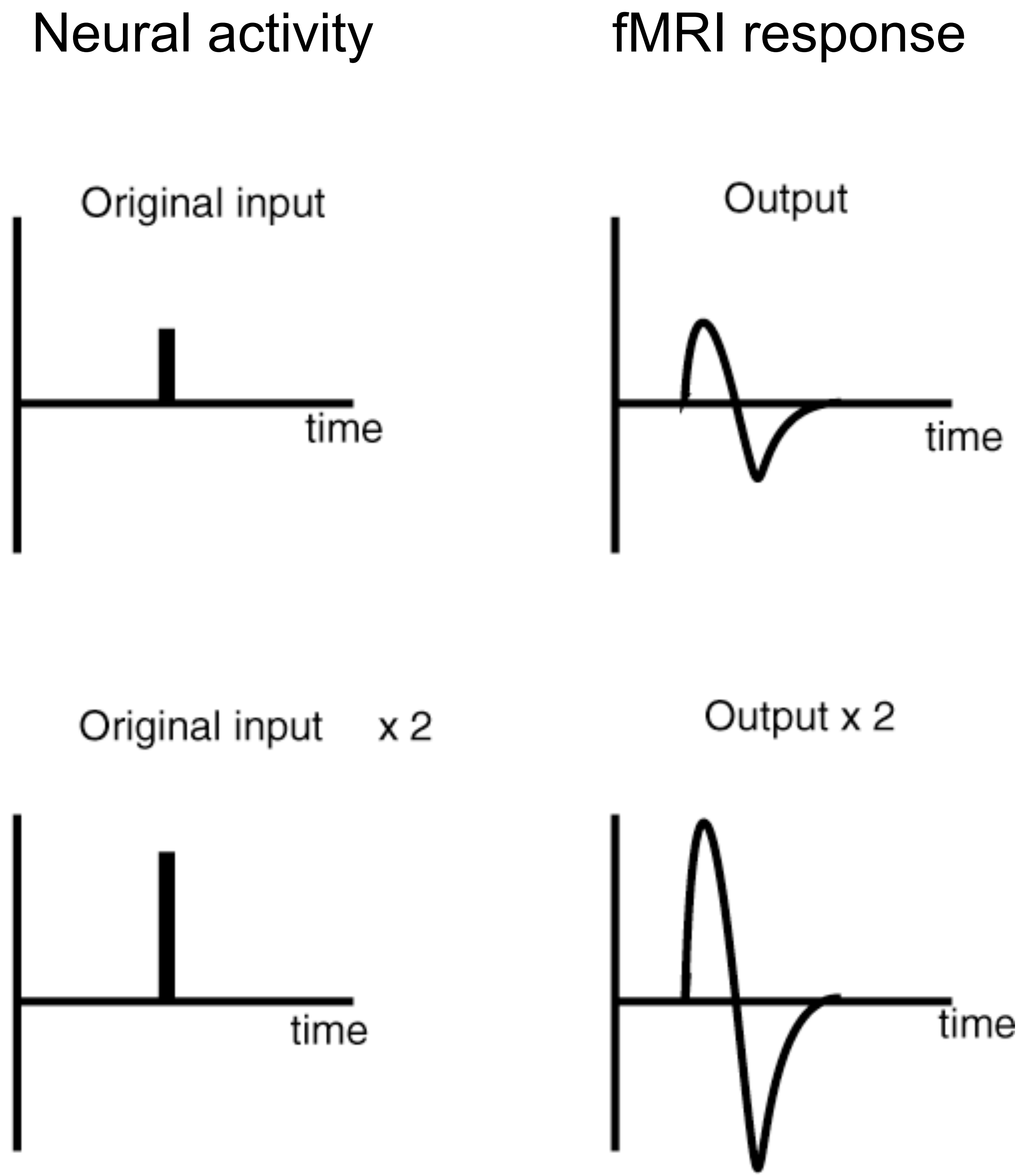
The Fourier transform is a useful change of basis for many signals



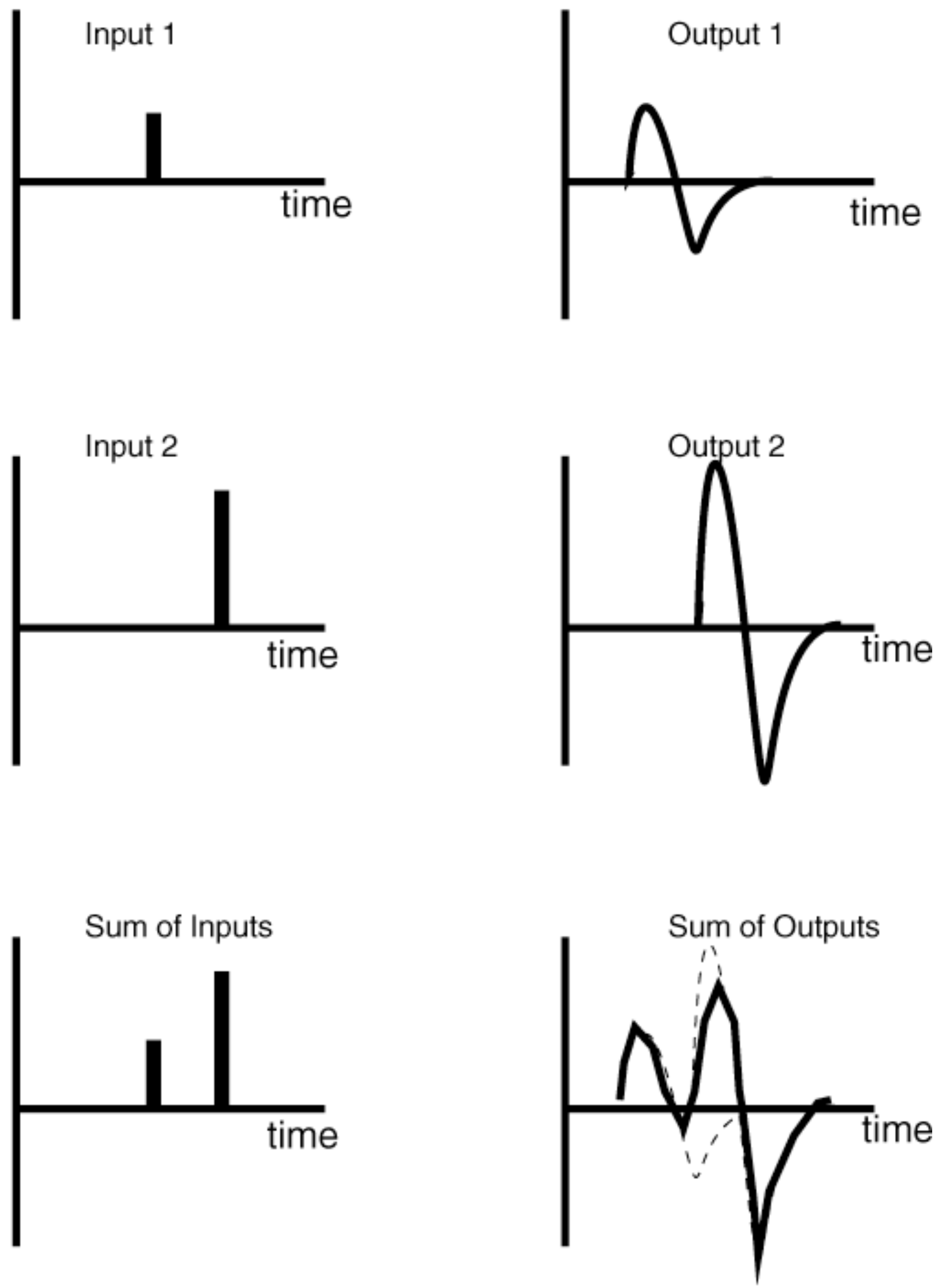
The Fourier transform is a useful change of basis for many signals

```
x = imread('cameraman.tif');
F = fft2(x); % Fourier transorm
n = 30; % Keep Fourier coefficients up to n
F(n+1:end-n+1, n+1:end-n+1) = 0; % zero out the rest
x2 = ifft2(F); % transform back to image domain

figure; colormap gray
subplot(2,2,1), imagesc(x), title('Original')
subplot(2,2,2), imagesc(x2), title('Remove most fourier components')
subplot(2,2,4), imagesc(log(abs(F))), title('Fourier transform of above')

% now zero out most of the pixels for comparison
x3 = x;
x3(n+1:end-n+1, n+1:end-n+1) = 0;

subplot(2,2,3), imagesc(x3), title('Remove most pixels')
```

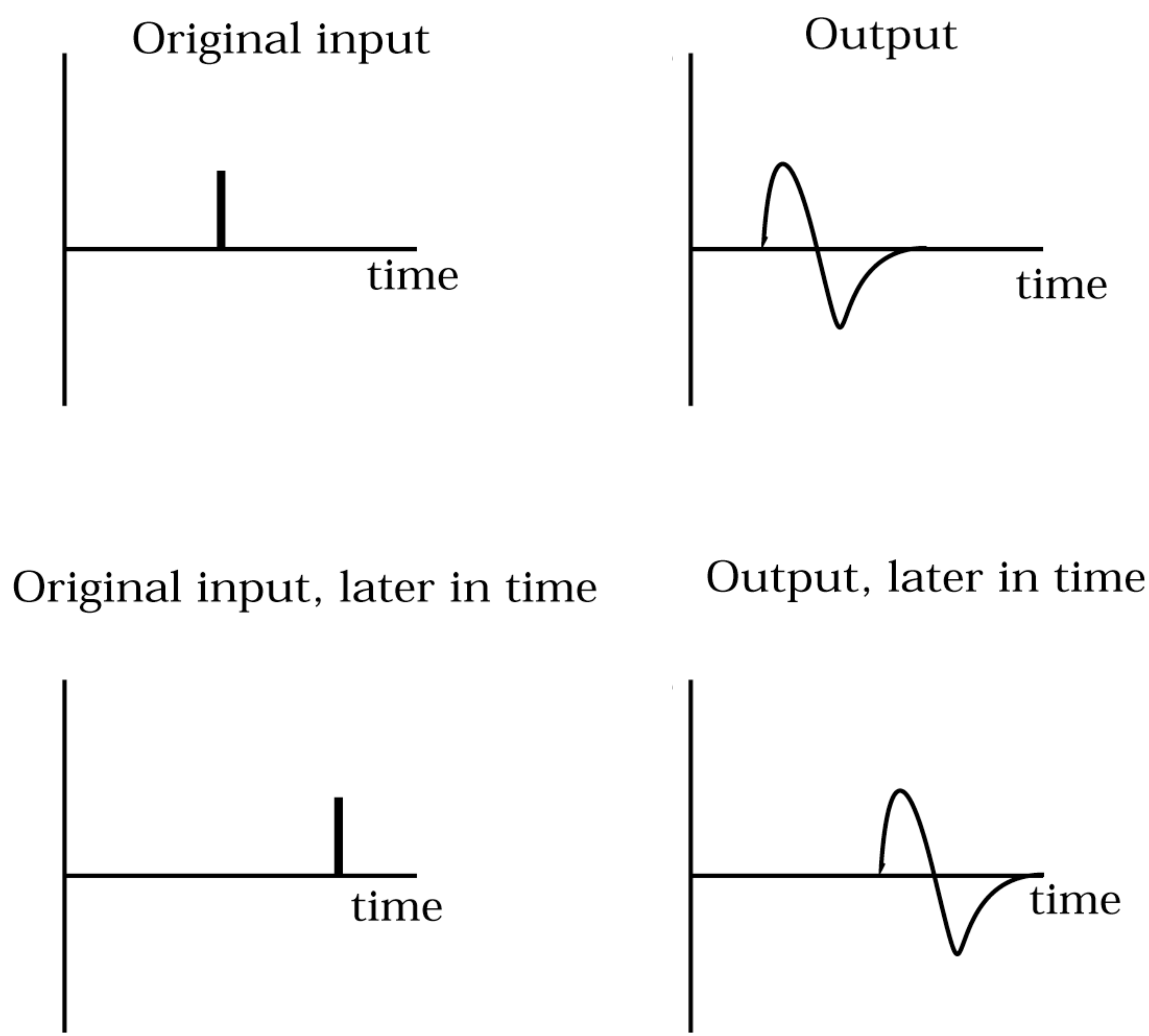
1. Homogeneity (scalar rule)



2. Additivity



Shift invariance



Linear systems

A system (or transform) converts (or maps) an input signal into an output signal:

$$y(t) = T[x(t)]$$

A linear system satisfies the following properties:

1) Homogeneity (scalar rule):

$$T[a\, x(t)] = a\, y(t)$$

2) Additivity:

$$T[x_1(t) + x_2(t)] = y_1(t) + y_2(t)$$

Often, these two properties are written together and called superposition:

$$T[a\, x_1(t) + b\, x_2(t)] = a\, y_1(t) + b\, y_2(t)$$

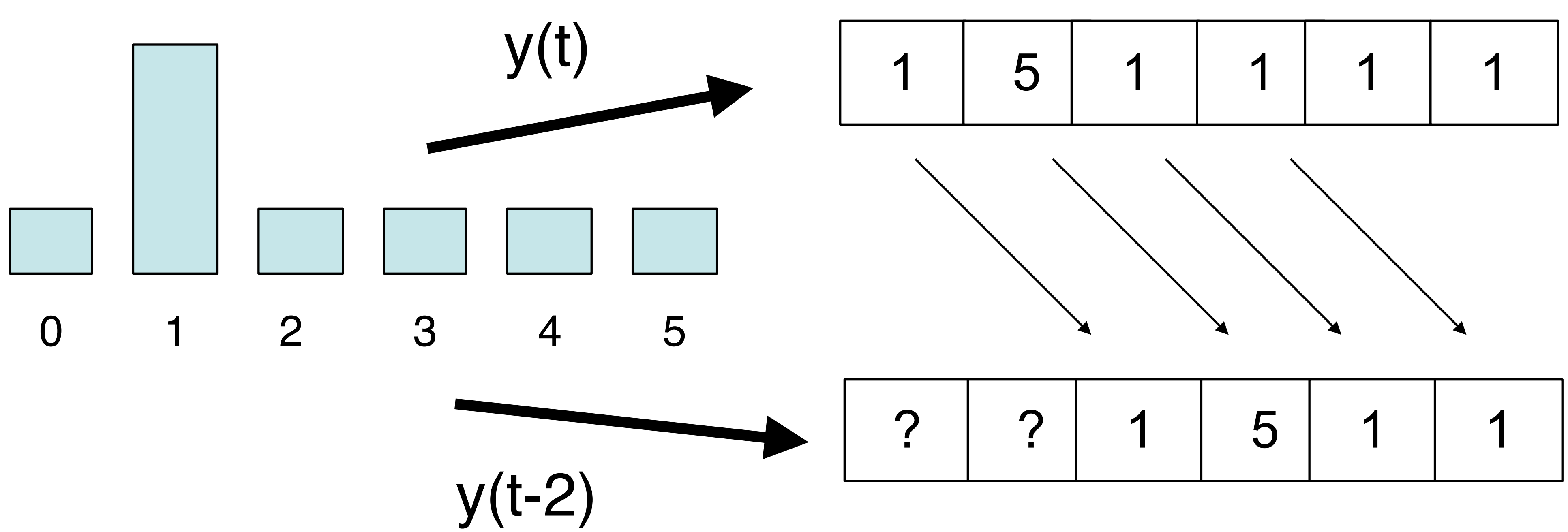
Shift invariance

For a system to be shift-invariant (or time-invariant) means that a time-shifted version of the input yields a time-shifted version of the output:

$$y(t) = T[x(t)]$$

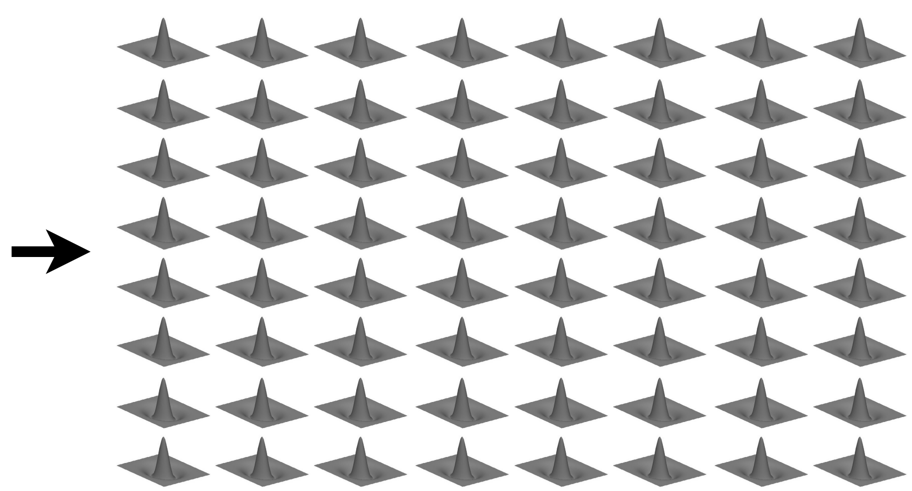
$$y(t - s) = T[x(t - s)]$$

The response $y(t - s)$ is identical to the response $y(t)$, except that it is shifted in time.



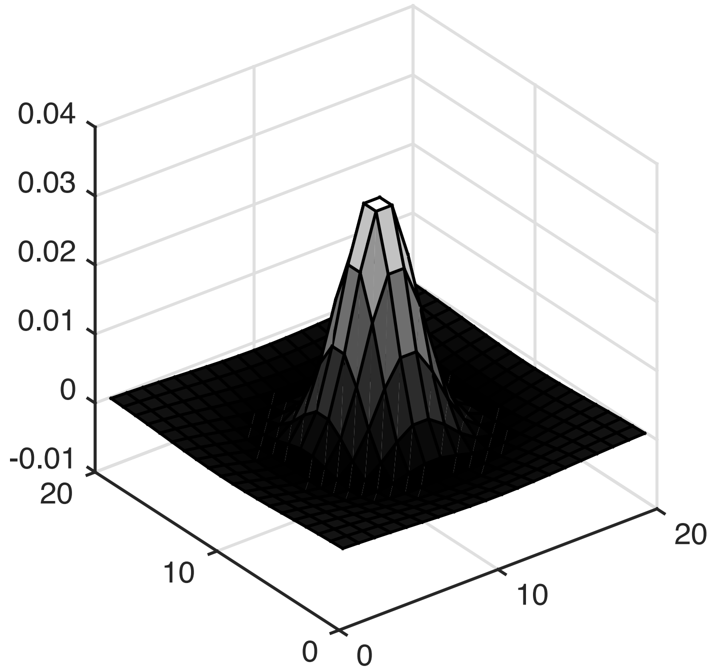
Neural Image - Reprise

A spatial receptive field may also be treated as a linear system, by assuming a dense collection of neurons with the same receptive field translated to different locations in the visual field. In this view, it is a linear, shift-invariant system.



See code on next slide

Neural Image - Reprise



```
% Make a Difference of Gaussian receptive field using fspecial
DoG = fspecial('gaussian', 20,2) -  fspecial('gaussian', 20,5);
im = imread('cameraman.tif');

% Make a neural image by convolution
neuralim = conv2(double(im), DoG);

% Show the image, the RF, and the neural image
figure,  subplot(1,3,1), imshow(im), subplot(1,3,2), surf(DoG)
subplot(1,3,3), imagesc(neuralim); colormap gray, axis image off
```

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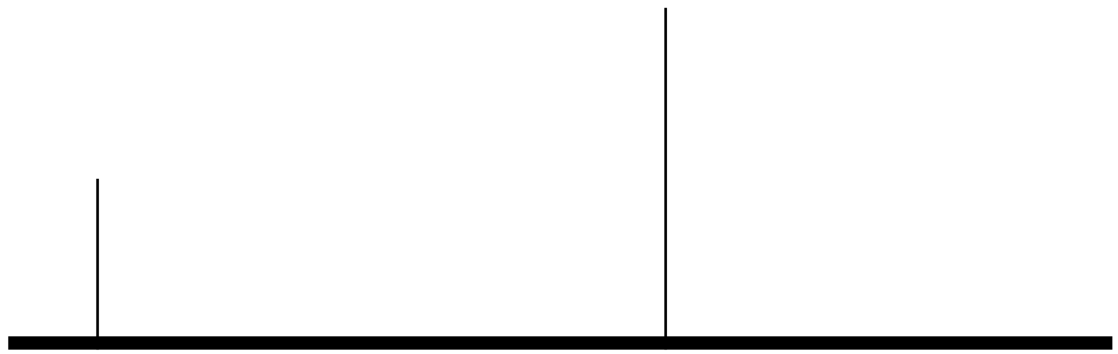
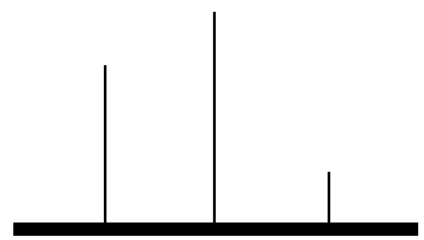
Convolution as sum of impulse responses

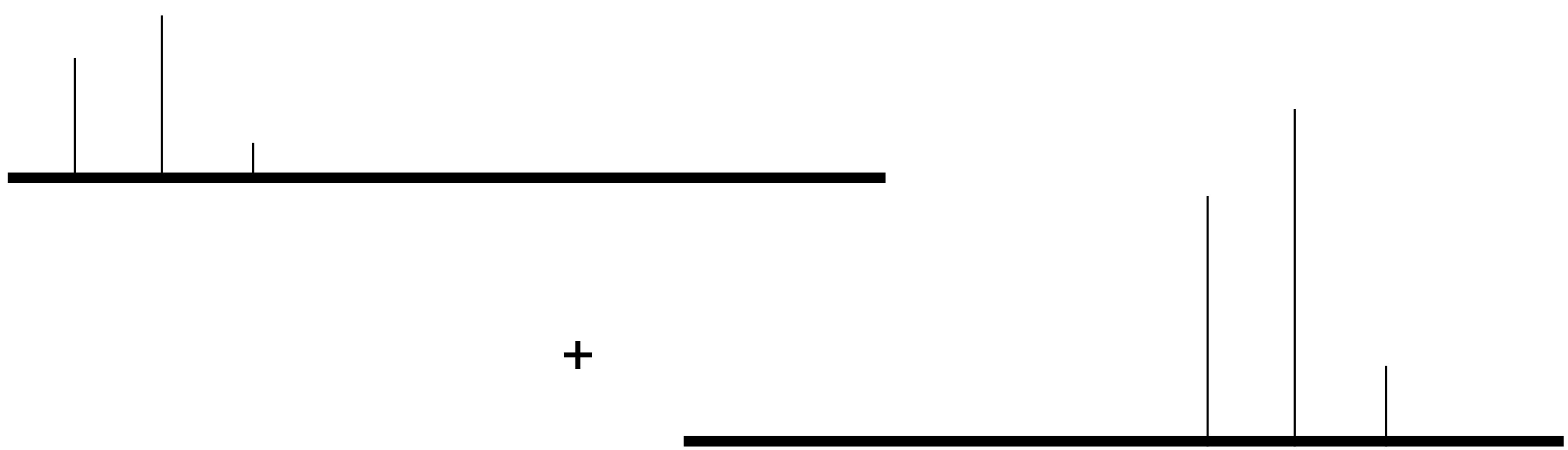
Input:



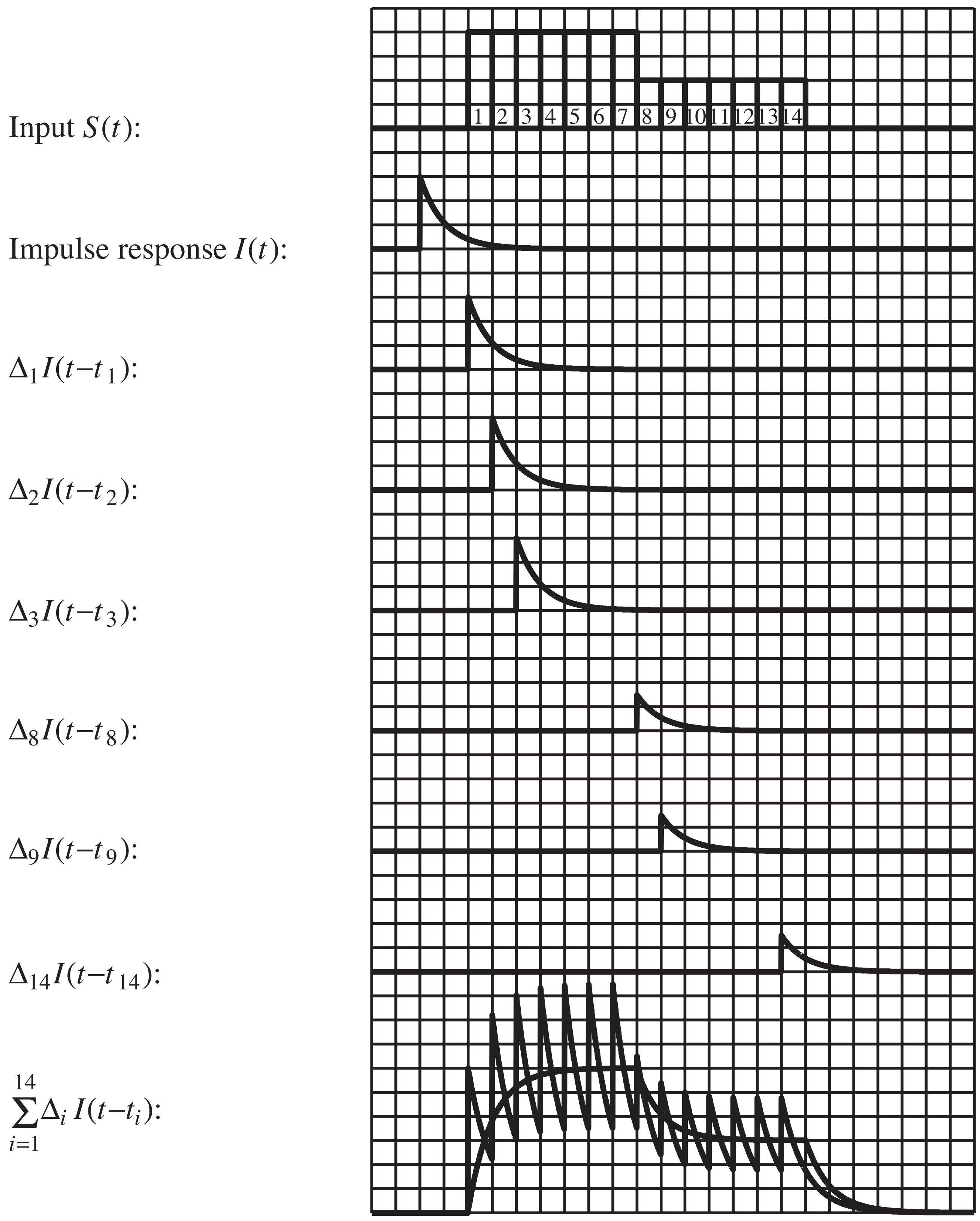
Impulse response:



Output:



Convolution as sum of impulse responses




```
% time vector
dt = 0.01;  T = 25;
t = dt:dt:T-dt;

% impulse response function
h = exp(-(t));

% signal
x = zeros(size(t));
x(t<19) = 2;
x(t<12) = 4;
x(t<5)  = 0;

% output of system by convolution
r = conv(x,h/sum(h), 'full');
r = r(1:length(t));

figure(1), clf,
plot(t,x, 'k', 'LineWidth', 4); hold on
plot(t,r, 'LineWidth', 4);

% Suppose our sampling was just one time point per
second
x2 = zeros(size(t));
x2(1:100:end) = x(1:100:end);

r2 = conv(x2,h, 'full');
r2 =r2(1:length(t));

stem(t,x2)
plot(t, r2, 'LineWidth', 4)
```

Convolution

Discrete-time signal: $x[n] = [x_1, x_2, x_3, \ldots]$

A system or transform maps an input signal into an output signal:

$$y[n] = \mathrm{T}\{x[n]\}$$

A shift-invariant, linear system can always be expressed as a convolution:

$$y[n] = \sum x[m]\, h[n-m]$$

where $h[n]$ is the impulse response.

Convolution derivation

Homogeneity:

$$T\{a\, x[n]\} = a\, T\{x[n]\}$$

Additivity:

$$T\{x_1[n] + x_2[n]\} = T\{x_1[n]\} + T\{x_2[n]\}$$

Superposition:

$$T\{a\, x_1[n] + b\, x_2[n]\} = a\, T\{x_1[n]\} + b\, T\{x_2[n]\}$$

Shift-invariance:

$$y[n] = T\{x[n]\} \Rightarrow y[n-m] = T\{x[n-m]\}$$

Convolution derivation (contd.)

Impulse sequence:

$d[n] = 1$ for $n = 0$, $d[n] = 0$ otherwise

Any sequence can be expressed as a sum of impulses:

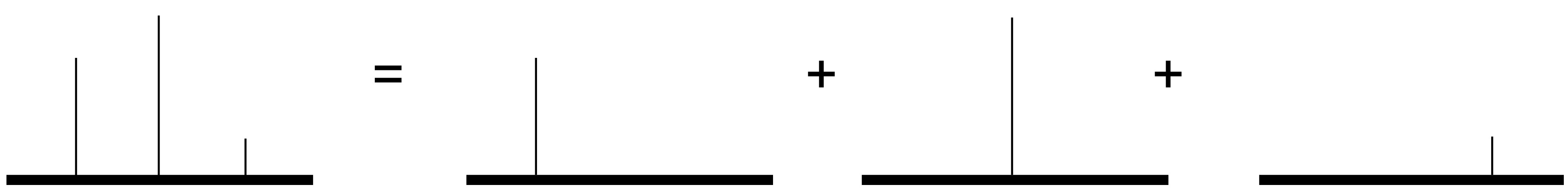
$$x[n] = \sum x[m]\, d[n-m]$$

where

$d[n-m]$ is impulse shifted to sample m

$x[m]$ is the height of that impulse

Example:



Convolution derivation (cont)

x[n]: input
y[n] = T{x[n]}: output
h[n] = T{d[n]}: impulse response

1) Represent input as sum of impulses:

$$y[n] = T\{x[n]\}$$

$$y[n] = T\left\{ \sum x[m]\, d[n-m] \right\}$$

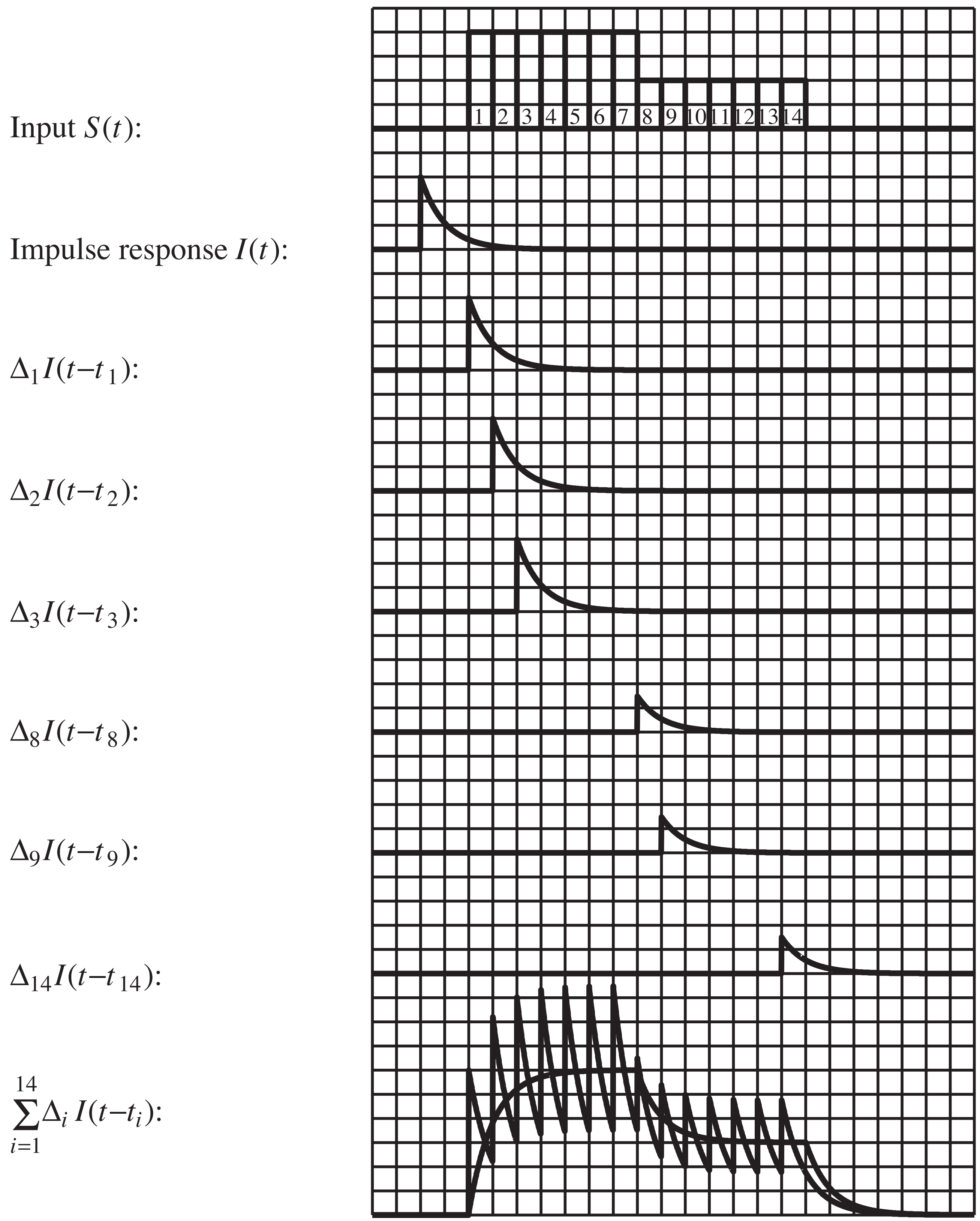
2) Use superposition:

$$y[n] = \sum x[m]\, T\{d[n-m]\}$$

3) Use shift-invariance:

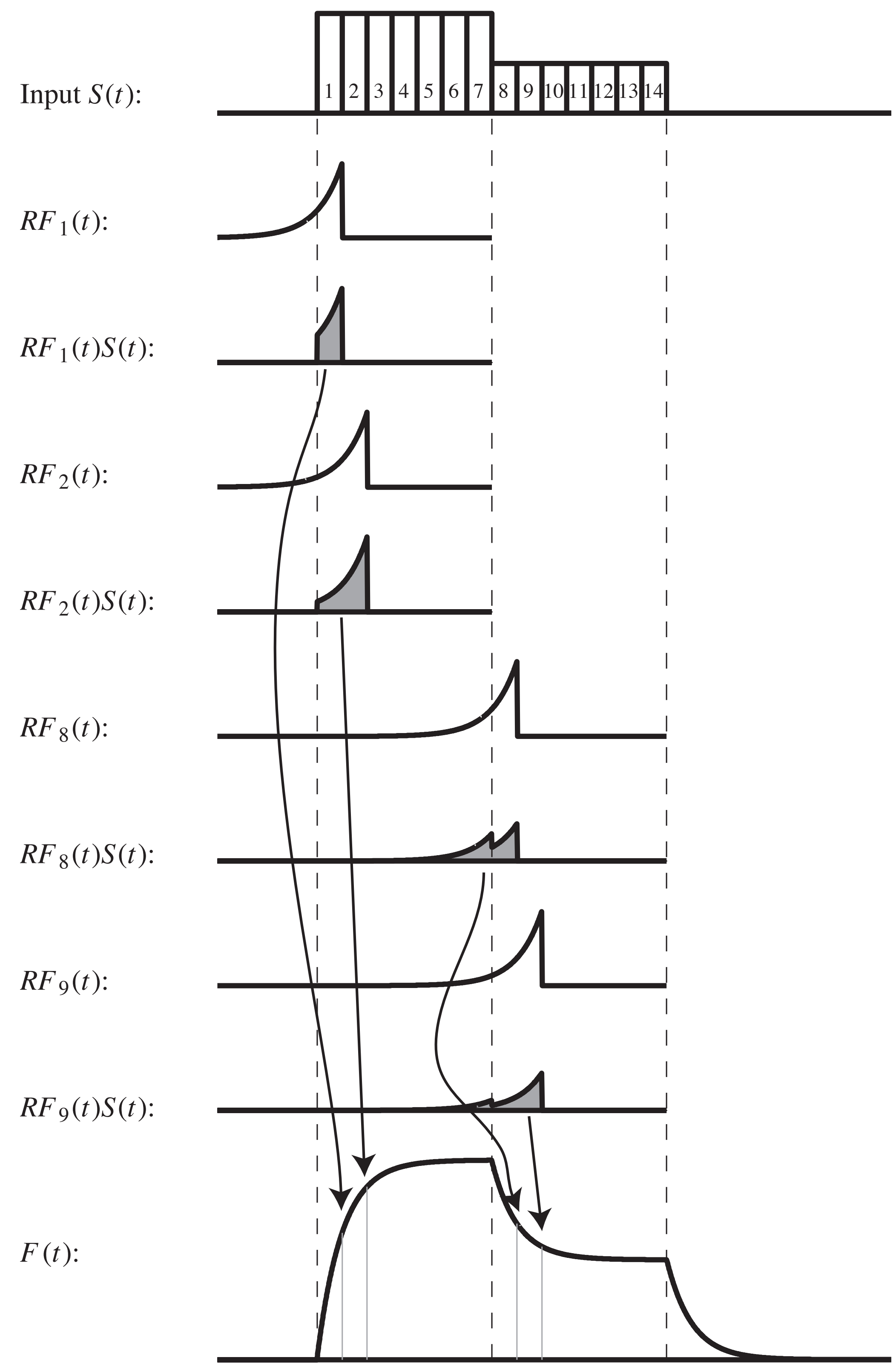
$$y[n] = \sum x[m]\, h[n-m]$$

Convolution as sum of impulse responses



Convolution as correlation with the "receptive field" (time-reversed impulse response):

```
% time
dt = 0.1; t = dt:dt:25;

% signal (input)
x = zeros(size(t));
x(t<19) = 2; x(t<12) = 4; x(t<5)  = 0;

% impulse response function
h = exp(-(t));

% convert impulse response function to RF by time reversing it
rf = flip(h)/sum(h);

% output
y = zeros(size(t));

% multiply time shifted RFs by (non-shifted) signal
figure(1); clf;

for ii = 1:length(t)
    % shift the receptive field by one time step
    rf = rf([end 1:end-1]);

    % correlate (multiply) receptive field with signal
    %   note that the signal does not shift; only the rf shifts.
    y(ii) = x*rf';

    % plot
    plot(t,x ,'k', t, rf*10,'r',t, y, 'b', 'LineWidth', 4);
    legend('signal', 'rf', ‘output’); drawnow();
end
```



Convolution as matrix multiplication

Linear system <=> matrix multiplication
Shift-invariant linear system <=> Toeplitz matrix

$$
\begin{pmatrix} \vdots \\ 5 \\ 2 \\ -3 \\ 4 \\ -6 \\ \vdots \end{pmatrix}
=
\begin{pmatrix}
 & & & & \vdots & & & & \\
\cdots & 1 & 2 & 3 & 0 & 0 & 0 & 0 & \cdots \\
 & 0 & 1 & 2 & 3 & 0 & 0 & 0 & \\
 & 0 & 0 & 1 & 2 & 3 & 0 & 0 & \\
 & 0 & 0 & 0 & 1 & 2 & 3 & 0 & \\
 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & \\
 & & & & \vdots & & & &
\end{pmatrix}
\begin{pmatrix} \vdots \\ 1 \\ 2 \\ 0 \\ 0 \\ -1 \\ 2 \\ -3 \\ \vdots \end{pmatrix}
$$

Columns contain shifted copies of the impulse response.
Rows contain time-reversed copies of impulse response.

Convolution as matrix multiplication

Linear system <=> matrix multiplication
Shift-invariant linear system <=> Toeplitz matrix

$$
\begin{pmatrix} \vdots \\ 5 \\ 2 \\ -3 \\ 4 \\ -6 \\ \vdots \end{pmatrix}
=
\begin{pmatrix}
 & & & \vdots & & & & \\
 & 1 & 2 & 3 & 0 & 0 & 0 & 0 & \\
 & 0 & 1 & 2 & 3 & 0 & 0 & 0 & \\
\cdots & 0 & 0 & 1 & 2 & 3 & 0 & 0 & \cdots \\
 & 0 & 0 & 0 & 1 & 2 & 3 & 0 & \\
 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & \\
 & & & & \vdots & & & &
\end{pmatrix}
\begin{pmatrix} \vdots \\ 1 \\ 2 \\ 0 \\ 0 \\ -1 \\ 2 \\ -3 \\ \vdots \end{pmatrix}
$$

Columns contain shifted copies of the impulse response.
Rows contain time-reversed copies of impulse response.

Convolution as sequence of weighted sums

past		present					future		
0	0	0	1	0	0	0	0	0	input (impulse)
	1/8	1/4	1/2	→					weights
0	0	0	1/2	1/4	1/8	0	0	0	output (impulse response)

0	0	0	1	1	1	1	1	1	input (step)
			1/8	1/4	1/2	→			weights
0	0	0	1/2	3/4	7/8	7/8	7/8	7/8	output (step response)

Continuous-time derivation of convolution

Pulses and impulses

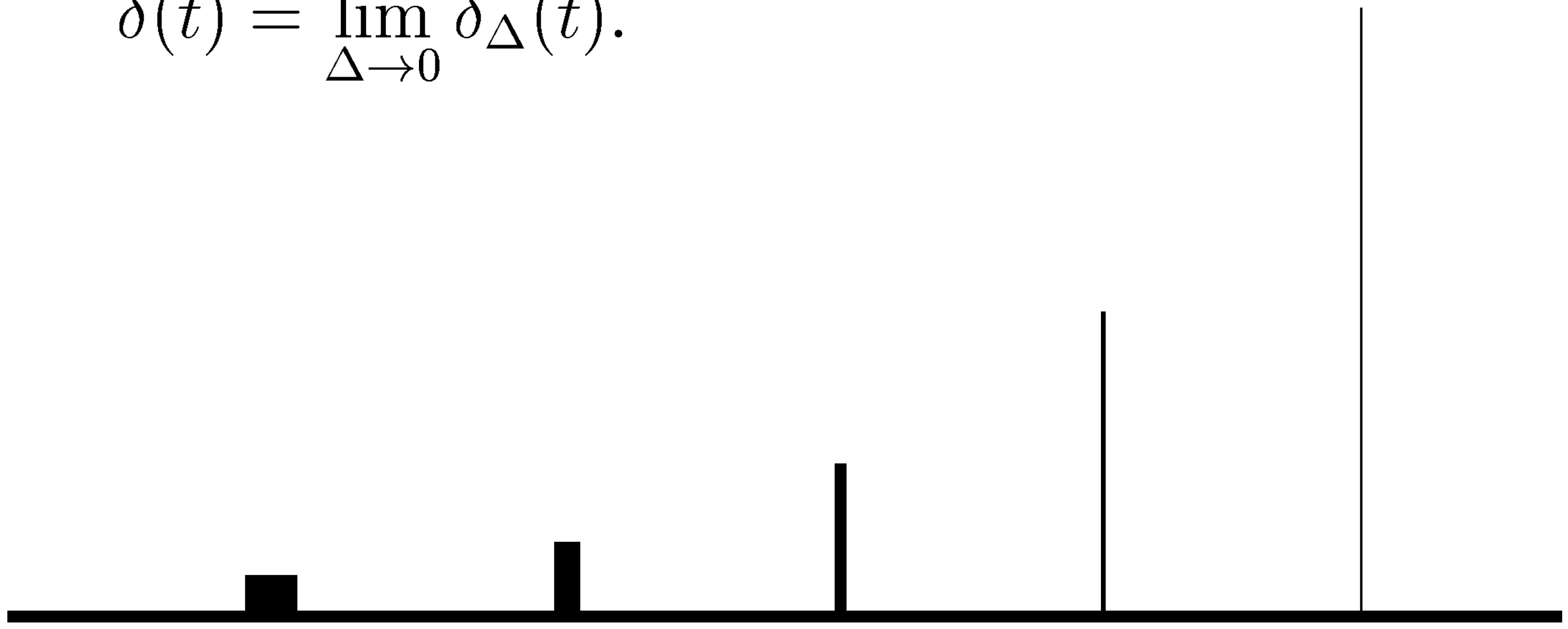
impulse

$$\delta(t) = \begin{cases} \infty & \text{if} \quad t = 0 \\ 0 & \text{otherwise} \end{cases}$$

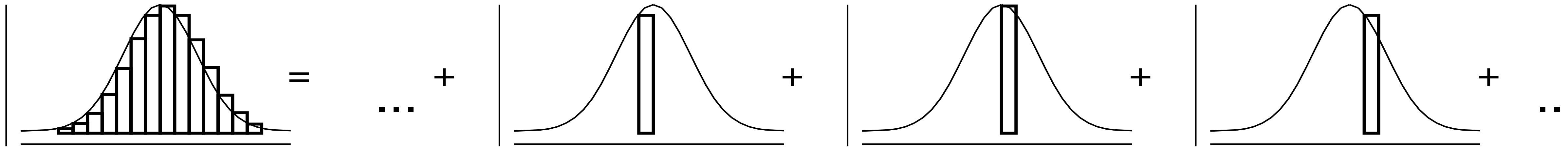
unit pulse

$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta} & \text{if} \quad 0 < t < \Delta \\ 0 & \text{otherwise} \end{cases}$$

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t).$$



Staircase approximation to continuous-time signal



$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\, \delta_\Delta(t - k\Delta)\, \Delta.$$

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\, \delta_\Delta(t - k\Delta)\, \Delta.$$

$$x(t) = \int_{-\infty}^{\infty} x(s)\, \delta(t - s)\, ds.$$

```
% time
t = dt:dt:10;

% signal (gaussian, centered at 5 with sd of 2)
x = exp(-(t-5).^2/2^2);

% discretely sample signal at d = 1 second steps
D = 1;

% show the sampling
figure(1), clf; hold on
for k = 1:10
    [~, whichTimePoint] = min(abs(t-k*D));
    plot(t,x, 'k', t, x(whichTimePoint) * (t-D*k < D & t-D*k>0), 'b');
    pause(.3)
end

% now do the same for finer sampling
D = 0.2;
figure(1), clf; hold on
for k = 1:10/D
    [~, whichTimePoint] = min(abs(t-k*D));
    plot(t,x, 'k', t, x(whichTimePoint) * (t-D*k < D & t-D*k>0), 'b');
    pause(.05)
end
```

Convolution

Representing the input signal as a sum of pulses:

$$\begin{aligned} y(t) = T[x(t)] &= T\left[\int_{-\infty}^{\infty} x(s)\,\delta(t-s)\,ds\right] \\ &= T\left[\lim_{\Delta\to 0}\sum_{k=-\infty}^{\infty} x(k\Delta)\,\delta_\Delta(t-k\Delta)\,\Delta\right]. \end{aligned}$$

Using additivity,

$$y(t) = \lim_{\Delta\to 0}\sum_{k=-\infty}^{\infty} T[x(k\Delta)\,\delta_\Delta(t-k\Delta)\,\Delta].$$

Taking the limit,

$$y(t) = \int_{-\infty}^{\infty} T[x(s)\,\delta(t-s)\,ds].$$

Using homogeneity (scalar rule),

$$y(t) = \int_{-\infty}^{\infty} x(s)\,T[\delta(t-s)]\,ds.$$

Defining $h(t)$ as the impulse response,

$$y(t) = \int_{-\infty}^{\infty} x(s)\,h(t-s)\,ds.$$

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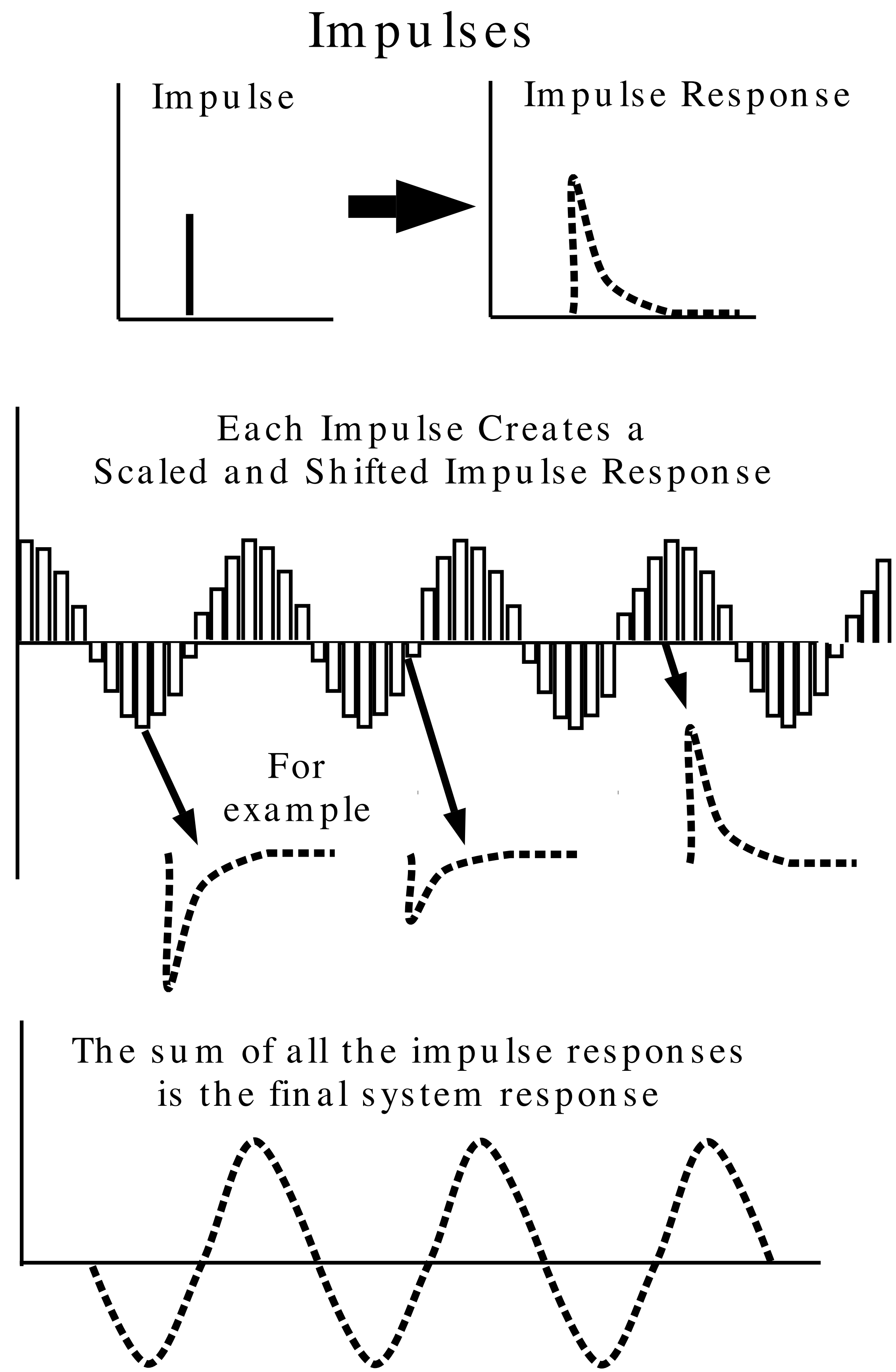
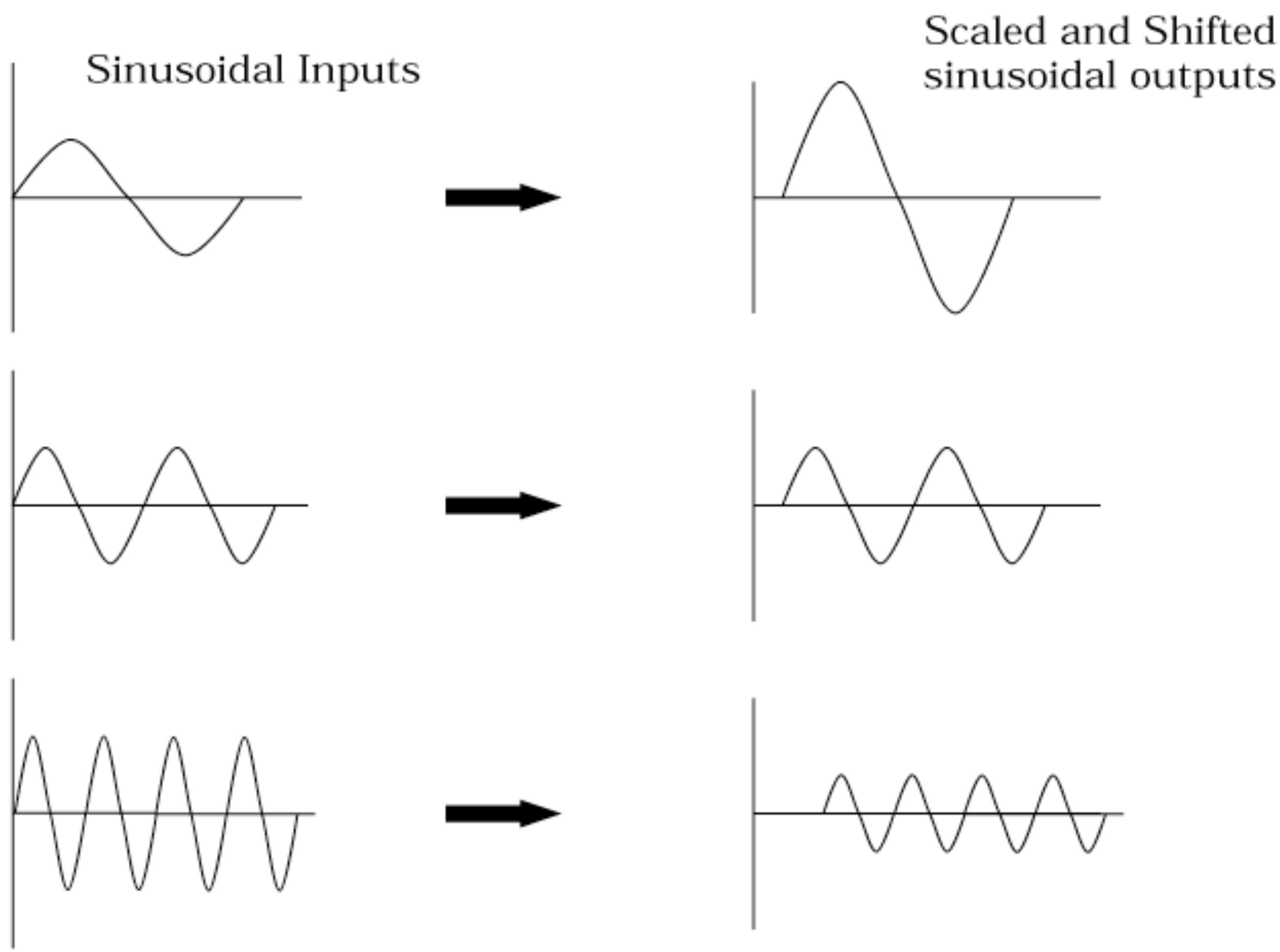
Shift-invariant linear systems and impulses



Shift-Invariant Linear Systems and Sinusoids

We measure the scaling and
shifting for each sinusoid



Shift-Invariant Linear Systems and Sinusoids