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COMPUTING THE JAGGEDNESS OF EDGES

Avi C. Naiman
Department of Computer Science, Hong Kong University of Science and Technology
Clear Water Bay, Kowloon, Hong Kong

Walter Makous
Center for Visual Science, University of Rochester
274 Meliora Hall, Rochester, NY 14627-0270, USA

Abstract

We address the properties of jaggies – the artifacts introduced by digital display devices in the representation of smooth, oblique edges – that might affect their visibility. We find that they have a hierarchical structure, and that the structure and the number of levels of the hierarchy can be computed if the slopes of the edges depicted are rational numbers.

Introduction

To display text and graphical objects on digital devices such as computer screens and printers, discrete approximations must be generated and presented. Quantization of analog representations may involve spatial, temporal, luminance and/or chromatic dimensions. For presentations intended for the human eye, visual artifacts that detract from the task to be performed must be identified and, if feasible, avoided or eliminated. Because the human visual system has finite resolving ability in each of these dimensions, we are interested in identifying the conditions under which visual artifacts arise and the attendant requirements to improve display technology.

One of the most commonly-recognized visual artifacts encountered in computer displays is often referred to as *jaggies*: the jagged spatial profile of oblique edges that are intended to be smooth. Previous work on jaggies has focused on mathematical models of edges as well as on algorithms for generating discrete approximations of smooth edges [Bresenham 65, Brons 74, Freeman 70, Hung 85]. We have recently embarked on a series of experiments to determine the visibility of jaggies [Naiman and Makous 94], addressing the question: Under what conditions can jagged edges be detected?

Our interest here lies primarily with high-contrast – i.e., black and white – edges. For present purposes, we restrict consideration to jagged edges that are specified by integral-coordinate endpoints corresponding to pixels. The slopes of such edges are rational numbers that can be specified by a pair of integers, which we refer to as n (numerator) and d (denominator). Without loss of generality, we further restrict our discussion to jagged edges in the first octant; i.e., $0 < (n, d)$ and $n \le d$. We assume that the display has a square grid of square pixels, the boundaries of which are aligned with the horizontal and vertical axes (as in an LCD display).

Composition of Jagged Edges

To introduce the concept of *jag level*, let us examine the jagged profiles of edges with a period of 16 pixels (i.e., $d = 16$). In Fig. 1-1, $n = 1$; i.e., there is one pixel of vertical displacement (VD) for every 16 pixels of horizontal displacement (HD). This regular pattern of jaggies is repeated for arbitrarily-long edges with a slope of 1/16. Notationally, we represent this as {16}, signifying that there are 16 HDs for each VD. We refer to '16' as an *element* of the *pattern* representing a slope of 1/16.

In Fig. 1-2, $n = 2$, with one VD for every 8 HDs. Notationally, this can be represented as {8, 8} or, more compactly, as $\{8^2\}$. The fact that there are two identical elements in this pattern – and no other elements – is indicative of the fact that 2/16 is a reducible, rational fraction. That is, the pattern of jaggies for one period of a 2/16-slope edge is identical to the pattern of two consecutive periods of a 1/8-slope edge. For subsequent computations, we must represent all slopes in their irreducible, rational fraction (IRF) form. This can be

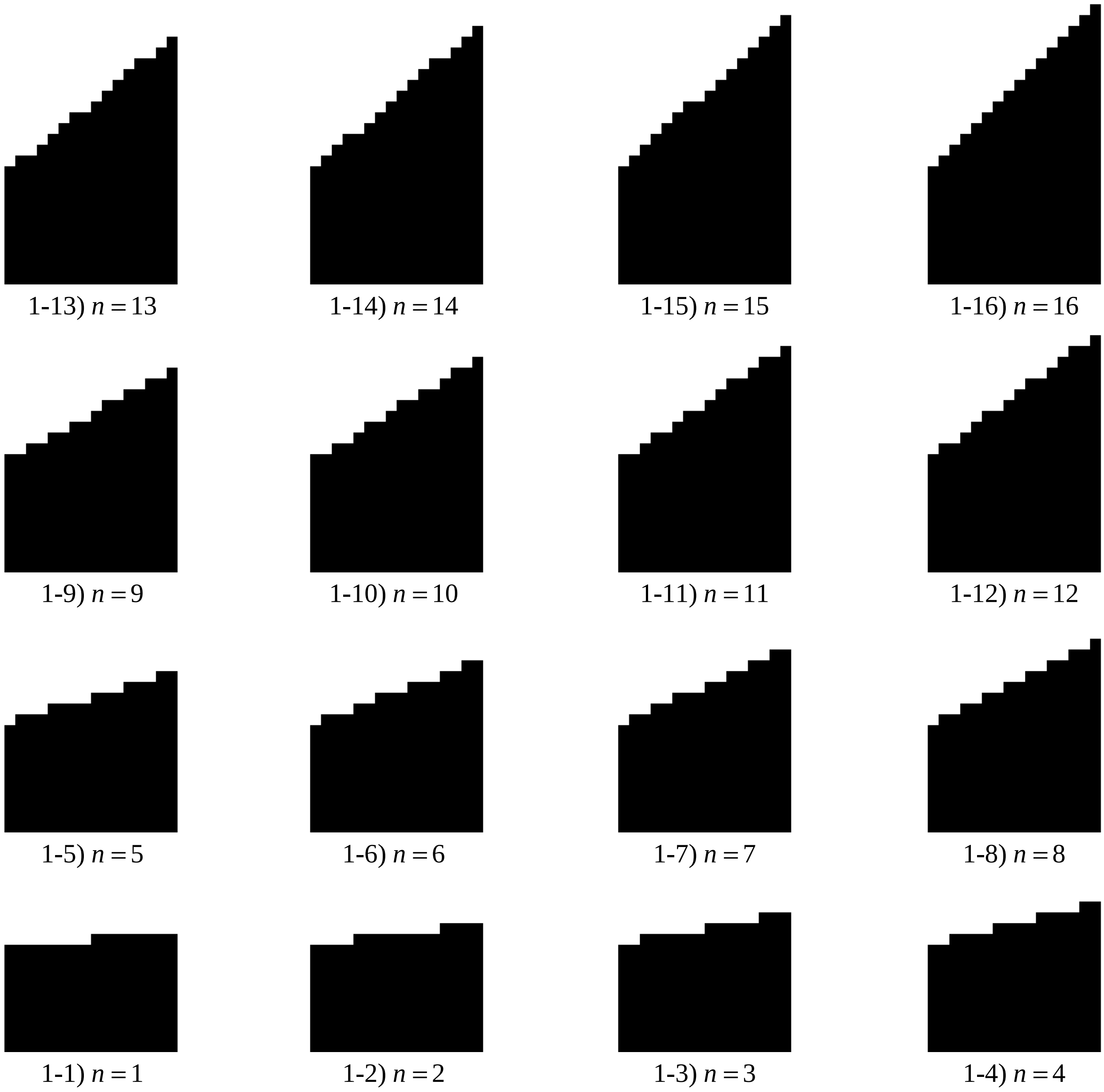

Figure 1: Jagged Edges. Jagged edge profiles are shown for $d = 16$ and $n = 1$ through 16.

achieved by dividing both the numerator and denominator by the greatest common divisor (GCD) of the two, using Euclid's algorithm [Graham *et al.* 89]: $n' = n / gcd(n, d)$ and $d' = d / gcd(n, d)$. Hence, for $n = 2$ and $d = 16$, $n' = 1$ and $d' = 8$, and the pattern of jaggies is represented by $\{8\}$.

In Figs. 1-4, 1-8 and 1-16, $n = 4$, 8 and 16, respectively, with one VD for every 4, 2 and 1 HD, respectively; after reduction, the patterns are represented as $\{4\}$, $\{2\}$ and $\{1\}$, respectively. In all these cases, there is an integral number of HDs for each VD. Consequently, the qualitative characteristic of regularity in the jaggies is maintained, and we refer to these patterns as having *first-level jags*.

Now let us consider a slope of $n = d - 1$. Examine the profile of this edge (for $n = 15$) in Fig. 1-15. As only $d - 1$ VDs must be made for d HDs, there is a single occurrence of a two-pixel HD, seen in the middle of the figure. In words, the profile pattern consists of 7 repetitions of 1 HD for every VD, followed by a single instance of 2 HDs for 1 VD, followed by another 7 repetitions of 1 HD for every VD. When more than one period of an edge with this slope is presented, the two end runs of 7 repetitions become consecutive, resulting in a pattern that consists of 14 repetitions of 1 HD for every VD, followed by a single instance of 2 HDs for 1 VD. In our notation, this is expressed as $\{1^{14}, 2\}$. Thus, the regular pattern that can be described as going up one and over one is interrupted at one location by going up two and over one. We regard any such interruption of a regular pattern of jags as a *higher-level* jag, in this case, a *second-level* jag. This second-level jag makes the jaggedness of the $n = d - 1$ profile much more visible than that of the $n = d$ profile [Naiman and Makous 94]. To appreciate this, the reader is invited to view the figures from a sufficient distance to notice that the jagged edge in Fig. 1-16 can be undetectable while the one in Fig. 1-15 is clearly visible.

In Fig. 1-14, $n = 14$, there are 2 two-pixel HDs, and, after reduction, the pattern is expressed as $\{1^6, 2\}$. In Fig. 1-12, $n = 12$, there are 4 two-pixel HDs, and the pattern is expressed as $\{1^3, 2\}$. Although the number of two-pixel HDs has changed in these last two instances, there is a regularity to the second-level jags superimposed on the first-level jags, as there was to the first-level jags superimposed on the zeroth-level jags (i.e., a horizontal line).

Now consider $n = 10$, as shown in Fig. 1-10. Here, there are 10 VDs made for every 16 HDs, and the profile pattern is $\{2^2, 1, 2, 1\}$, which is repeated a second time within the period. Without the initial 2 in the sequence, the pattern would be $\{2, 1\}$ (repeated a second time), which consists entirely of first- and second-level jags. However, the initial 2 interrupts this regular pattern, and this interruption represents a *third-level* jag. The remainder of Fig. 1 shows the jagged edge patterns for $d = 16$ and the other possible values for n; Table 1 lists the corresponding jag levels, as computed in the next section.

Note that at each successive level of the hierarchy, the maximum density of the jags at that level decreases. This means that the frequency of the fundamental of the Fourier spectrum of jags at higher levels of the hierarchy also decreases. As lower frequencies are more visible than higher frequencies [Hamerly and Springer 81], the highest jag level in an edge is more visible than any of the lower levels. Hence, to determine the visibility of an edge's jaggedness [Naiman and Makous 94], we compute the highest jag level present.

Computing Jag Levels

For a particular n and d, the jag level of the edge profile can be computed by iteratively removing higher-order jag levels, until only a first-order jag remains. In describing our approach to computing jag levels, we use the term *singly-occurring* to refer to an element in a pattern that is neither preceded nor followed by the same element.

Discretization patterns that correspond to straight edges are known to obey the following 'straightness properties' [Brons 74, Freeman 70, Hung 85]:

1) there are, at most, two different elements in a pattern;
2) if there are two different elements, they differ by a value of 1;
3) at least one of the two different elements always occurs singly;
4) the *distribution* of the elements that do not occur singly obeys these straightness properties.

This final property implies a hierarchical uniformity to the distribution of jags, at all jag levels.

We illustrate these straightness properties with reference to entry 13 in Table 1, $\{1^4, 2, 1^3, 2, 1^3, 2\}$:

1) there are two different elements in the pattern, '1' and '2';
2) the elements differ by a value of 1;
3) the element '2' always occurs singly (hence, we do not accompany it with an exponent);

n	slope	IRF	Pattern Iterations			Jag
			1	2	3	Level
1	1/16	1/16	16			1
2	2/16	1/8	8			1
3	3/16	3/16	5^2, 6	2		2
4	4/16	1/4	4			1
5	5/16	5/16	3^4, 4	4		2
6	6/16	3/8	3^2, 2	2		2
7	7/16	7/16	2^3, 3, 2^2, 3	3^1, 2	1	3
8	8/16	1/2	2			1
9	9/16	9/16	1, 2^4, 1, 2^3	4^1, 3	1	3
10	10/16	5/8	1, 2^1, 1, 2^2	1^1, 2	1	3
11	11/16	11/16	1^2, 2, 1^1, 2, 1^1, 2, 1^1, 2, 1^1, 2	2, 1^4	4	3
12	12/16	3/4	1^2, 2	2		2
13	13/16	13/16	1^4, 2, 1^3, 2, 1^3, 2	4, 3^2	2	3
14	14/16	7/8	1^6, 2	6		2
15	15/16	15/16	1^{14}, 2	14		2
16	16/16	1/1	1			1

Table 1: Computing Jag Level. Pattern iterations and jag levels are listed for $d = 16$ and $n = 1$ through 16.

4) the distribution of the ‘1’ elements is $\{4, 3^2\}$ and it obeys the straightness properties:
   1) the two elements are ‘3’ and ‘4’;
   2) they differ by 1;
   3) the element ‘4’ occurs singly;
   4) the distribution of the ‘3’ elements is {2}, which is a first-level jag.

Consequently, there are a total of 3 levels of jags in an edge with a slope of 13/16.

The remainder of Table 1 shows the iterations for all possible n, when $d = 16$. To assist in following the iterations, we have included exponents of 1 for elements that do not occur singly elsewhere in the pattern. In this manner, successive iterations simply consist of converting the exponents of the element that does not occur singly into the next pattern (replacing repetitions of these elements with an exponential representation). Note that, when there are only two elements in a pattern and they both occur singly (e.g., iteration 2 for $n = 7$, 9 and 10), it makes no difference which element is considered as occurring singly: the next iteration will always be {1}.

The algorithm presented by Brons for generating patterns corresponding to IRF-represented slopes [Brons 74] provides the final ingredient needed in an iterative process for determining the jag level of any edge with a rational slope specified by n and d:

```
compute IRF form: n′/d′
generate pattern for n′/d′
set jag level to 1
while there are 2 or more elements in pattern
    replace pattern with exponents in pattern
    represent new pattern in exponent form
    increase jag level by 1
```

Upon termination, the correct jag level will have been set.

Conclusions

The jaggedness of oblique edges represented by digital display devices follows a hierarchical structure. For slopes that are rational numbers, an iterative algorithm was presented that calculates the structure and the number of levels of the hierarchy.

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