Bayesiansk statistik – ett alternativ till t-test och ANOVA?

Uppsala
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NB: If you haven’t filled out the questionnaire yet, please do so!
(for link: see tutorial announcement email)
Bayesian statistics #1: Hypothesis testing

Somewhere in a digital cloud
17 June 2020

Ronald van den Berg
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Tutorial #1: hypothesis testing

Examples of hypothesis testing:

• Is drug $D$ more effective than a placebo?

• Is there a correlation between age and mortality rate in disease $Y$?

• Does model $A$ fit the data better than model $B$?

• Do my subjects have a non-zero guessing rate?
Tutorial #2 (next week): hypothesis testing

Examples of estimation:

- On what percentage of people is this drug effective?
- How strong is the correlation between age and mortality rate in disease Y?
- How much better does model A fit the data than model B?
- How frequently did subjects guess in my experiment?
Why use statistics?
Why do we need statistical tests?

Differences are probably due to random variation

Differences are probably due to an effect of group
Why do we need statistical tests?

Task of statistics is to quantify this "probably"

Differences are probably due to random variation

Differences are probably due to an effect of group
Is there an effect of group on performance?

H0: There is no effect of group on performance
H1: There is an effect of group on performance
Is there an effect of group on performance?

**Frequentist approach**
Compute $p($extremeness of the data $| H0$ is true$)$

**Bayesian approach**
Compute $p($data $| H0$ is true$) / p($data $| H1$ is true$)$

H0: There is no effect of group on performance
H1: There is an effect of group on performance
Frequentist approach
There are two major schools of frequentist stats vs. [90x121]The presently standard approach to hypothesis testing is an inconsistent hybrid that every decent statistician would reject (Gigerenzer, 2004).
Hypothesis testing: Fisher's approach

1. Formulate a null hypothesis, $H_0$
   E.g.: “the drug has no effect on recovery speed”

2. Compute $p$
Hypothesis testing: Fisher's approach

1. Formulate a null hypothesis, $H_0$
   E.g.: “the drug has no effect on recovery speed”

2. Compute $p$, i.e., the probability of observing your data or more extreme data if $H_0$ were true

3. A low $p$ value implies that either something rare has occurred or $H_0$ is not true

   - **Power analysis** has no place in this framework
   - **High $p$** does not mean to accept $H_0$

**Reasoning:**
the lower $p$, the more certain we can be that $H0$ is false

- sounds reasonable, but ultimately a flawed way to test hypotheses
A p-roblem
Applying Fisher's approach to the case of Sally Clark

1996: Clark’s **1st son died** a few weeks after birth (SIDS?)
1998: Clark’s **2nd son died** a few weeks after birth (SIDS again????)
1999: Clark was **found guilty of murder** and given two life sentences

The conviction was partly based on the following statistical argument:

- H0: babies died from "Sudden Infant Death Syndrome" (SIDS) aka "crib death"
- SIDS occurrence rate is 1 in 8,500
- The chance of this happening twice is 1 in 73 million, i.e., \( p = 0.0000000137 \)
- Therefore, H0 is rejected
- Therefore, she must be guilty (double murder)

**What is wrong with this line of reasoning?**
Applying Fisher's approach to the case of Sally Clark

Even though H0 is unlikely, other hypotheses may be even more unlikely!!

The conviction was partly based on the following statistical argument:

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What is wrong with this line of reasoning?
Applying Fisher's approach to the case of Sally Clark

Evidence is best treated as a relative concept

“How (im)probable is H0, relative to H1?”

- H1: double murder
- Infant murder rate in UK: approximately 1 in 33,000 (*)
- The chance of this happening twice is 1 in 1.1 billion, i.e., p = 0.000000000918
- SIDS is 15 times more likely than murder!

Applying Fisher's approach to the case of Sally Clark

How did it end for Clark?

• 1996: Clark's **first son died** suddenly within a few weeks of his birth
• 1998: Clark's **second son died** suddenly within a few weeks of his birth
• 1999: Clark was **found guilty of murder** and given two life sentences
• 2003: Clark is set free, yet highly traumatized
• 2007: Clark dies from alcohol poisoning
Applying Fisher's approach to the case of Sally Clark

The same kind of flawed reasoning was part of Lucia de Berk’s conviction in the Netherlands
The deeper problem here:

• Some events are unlikely under *any* hypothesis
The deeper problem here:

- Some events are unlikely under any hypothesis
- Should we then reject them all and consider the event unexplainable?

**Solution:** lower the $\alpha$ value for rare events?

... no scientific worker has a fixed level of significance at which from year to year, and in all circumstances, he rejects hypotheses; he rather gives his mind to each particular case in the light of his evidence and his ideas.

Sir Ronald A. Fisher (1956)
The deeper problem here:

- Some events are unlikely under *any* hypothesis
- Should we then reject them all and consider the event unexplainable?

**Solution:** lower the $\alpha$ value for rare events?

However: how to do this without knowing the cause of the event??
The Bayes factor
Introduction to the Bayes Factor

\[
p\left(H_0 \mid D\right) \quad \text{Probability of Hypothesis 0, given the data}
\]

\[
\frac{p\left(H_0 \mid D\right)}{p\left(H_1 \mid D\right)} \quad \text{Probability of Hypothesis 1, given the data}
\]
Introduction to the Bayes Factor

\[
\frac{p(H_0 \mid D)}{p(H_1 \mid D)} = \frac{p(D \mid H_0)}{p(D \mid H_1)} \times \frac{p(H_0)}{p(H_1)}
\]

Posterior ratio

Prior ratio

Bayes factor

Indicates how many times more likely the data are under H0 compared to H1
Introduction to the Bayes Factor

By definition a *relative* measure

Easy, pleasant interpretation(s)

Allows to quantify evidence in favor of the null!

Generalizes more easily than frequentist approach?

\[
\frac{p(H_0|D)}{p(H_1|D)} = \frac{p(D|H_0)}{p(D|H_1)} \times \frac{p(H_0)}{p(H_1)}
\]

Posterior ratio

Prior ratio

Bayes factor

Alternative interpretation:

BF indicates the change from prior odds to posterior odds brought about by the data
Introduction to the Bayes Factor

\[
\frac{p(H_0|D)}{p(H_1|D)} = \frac{p(D|H_0)}{p(D|H_1)} \times \frac{p(H_0)}{p(H_1)}
\]

Visual interpretation of the Bayes factor:

\[BF_{10} = 1\]
Introduction to the Bayes Factor

\[
\frac{p(H_0 | D)}{p(H_1 | D)} = \frac{p(D | H_0)}{p(D | H_1)} \times \frac{p(H_0)}{p(H_1)}
\]

- **Posterior ratio**
- **Bayes factor**
- **Prior ratio**

**Visual interpretation of the Bayes factor**

\[
BF_{10} = \frac{1}{3}
\]

\[
BF_{01} = 3
\]
Introduction to the Bayes Factor

$$\frac{p(H_0 \mid D)}{p(H_1 \mid D)} = \frac{p(D \mid H_0)}{p(D \mid H_1)} \times \frac{p(H_0)}{p(H_1)}$$

Posterior ratio  
Prior ratio  
Bayes factor

Visual interpretation of the Bayes factor

$$BF_{10} = 3$$
$$BF_{01} = \frac{1}{3}$$
Guideline for interpreting BF evidence strength
(source: Wagenmakers et al. 2016)

<table>
<thead>
<tr>
<th>Bayes factor, $BF_{10}$</th>
<th>Evidence category</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&gt; 100$</td>
<td>Extreme evidence for $\mathcal{H}_1$</td>
</tr>
<tr>
<td>30 - 100</td>
<td>Very strong evidence for $\mathcal{H}_1$</td>
</tr>
<tr>
<td>10 - 30</td>
<td>Strong evidence for $\mathcal{H}_1$</td>
</tr>
<tr>
<td>3 - 10</td>
<td>Moderate evidence for $\mathcal{H}_1$</td>
</tr>
<tr>
<td>1 - 3</td>
<td>Anecdotal evidence for $\mathcal{H}_1$</td>
</tr>
<tr>
<td>1</td>
<td>No evidence</td>
</tr>
<tr>
<td>1/3 - 1</td>
<td>Anecdotal evidence for $\mathcal{H}_0$</td>
</tr>
<tr>
<td>1/10 - 1/3</td>
<td>Moderate evidence for $\mathcal{H}_0$</td>
</tr>
<tr>
<td>1/30 - 1/10</td>
<td>Strong evidence for $\mathcal{H}_0$</td>
</tr>
<tr>
<td>1/100 - 1/30</td>
<td>Very strong evidence for $\mathcal{H}_0$</td>
</tr>
<tr>
<td>$&lt; 1/100$</td>
<td>Extreme evidence for $\mathcal{H}_0$</td>
</tr>
</tbody>
</table>
### The two approaches in 5 steps

<table>
<thead>
<tr>
<th>Frequentist approach (Fisher)</th>
<th>Bayesian approach</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Step 1</strong></td>
<td>Formulate a <strong>single</strong> hypothesis H0</td>
</tr>
<tr>
<td><strong>Step 2</strong></td>
<td>Decide on <strong>all</strong> study factors <strong>before</strong> measuring a single data point (sample size, what to do with outliers, etc) – <strong>revising these decisions later would invalidate the test</strong></td>
</tr>
<tr>
<td><strong>Step 3</strong></td>
<td>Gather data</td>
</tr>
<tr>
<td><strong>Step 4</strong></td>
<td>Compute $p$</td>
</tr>
<tr>
<td><strong>Step 5</strong></td>
<td>If $p &lt; 0.05$: reject H0 If $p &gt; 0.05$: conclude nothing</td>
</tr>
</tbody>
</table>
Fisherman vs Bayesian statistics:

- **p value**
  - Evidence is absolute (about single hypothesis)
  - Can only reject hypotheses
  - Tests are problem-specific
  - Confusing for non-statisticians

- **Bayes factor**
  - Evidence is always relative (w.r.t. alternative hypotheses)
  - Can reject and support hypotheses
  - Tests are general
  - Much less confusing
Fisherian vs Bayesian statistics:

**p value**
- Evidence is absolute (about single hypothesis)
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Fisherian vs Bayesian statistics:

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- Evidence is *absolute* (about single hypothesis)
- Can only *reject* hypotheses
- Tests are problem-specific
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Fisherian vs Bayesian statistics:

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- Evidence is **absolute** (about single hypothesis)
- Can only **reject** hypotheses
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Why isn’t everyone a Bayesian???
Fisherian vs Bayesian statistics:

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**Bayes factor**
- Evidence is always **relative** (w.r.t. alternative hypotheses)
- Can **reject and support** hypotheses
- Tests are general?
- Less confusing?
- **Computationally expensive**
Fisherian vs Bayesian statistics:

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- Less confusing?
- Computationally expensive
- Requires specification of **priors**
Fisherian vs Bayesian statistics:

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- Tests are general?
- Less confusing?
- Computationally expensive
- Requires specification of **priors**

“Objective”

“Subjective”
Different philosophies

**Bayesians** quantify degrees of belief
-> highly subjective

**Frequentists** quantify long-term frequencies
-> claimed to be fully objective
Example #1:

Correlation analysis
Correlation - example

Two common questions:
1. Is the correlation "real"?
2. What is a plausible estimate of the strength of the “true” correlation?

Frequentist approach:
• Assume that data comes from a bivariate normal distribution
• Compute p value to answer first question
• Compute confidence interval to answer second question
Intuitive way to think about the p-value:

\[ p \approx \text{probability of finding } r_{\text{sample}} > 0.39 \text{ if } r_{\text{population}} = 0 \]

Formally, however

1. Compute t-statistic

\[ t^* = \frac{r \sqrt{n - 2}}{\sqrt{1 - r^2}} \]

2. Compute \( p = p(t^* > 0.39 \mid r_{\text{population}} = 0) \)

Underlying logic:
If \( r_{\text{population}} = 0 \), then \( t^* \) follows a t distribution with \( n-2 \) degrees of freedom
Correlation – frequentist results

H0: No correlation between height ratio and relative support

Frequentist results:
• \( p = 0.007 \)
• \( CI = [0.12; 0.62] \)

What have we learned from this analysis?

1. If the “true” (population-level) correlation were 0, we would have only 0.7% chance of finding data as extreme as our sample

2. We can be 95% confident that the “true” correlation is between 0.12 and 0.62

Wrong! This is a Bayesian interpretation of a frequentist concept!
Correlation analysis: a Bayesian approach
Bayesian correlation test

Same assumption
The data come from a bivariate normal distribution

Same question
Is there any evidence for a correlation at population level?

Different way to quantify this evidence
- Bayes factor instead of $p$ value
- Credible interval instead of confidence interval
In the context of correlation analysis, we define:

$H_0: r = 0$

$H_1: r \neq 0$

Hence, we want to compute

$$BF_{01} = \frac{p(D | r = 0)}{p(D | r \neq 0)}$$
Bayesian correlation test

\[ BF_{01} = \frac{p(x, y \mid r = 0)}{p(x, y \mid r \neq 0)} \]

Hence, we want to compute

\[ BF_{01} = \frac{p(D \mid H_0)}{p(D \mid H_1)} = \frac{p(x, y \mid r = 0)}{p(x, y \mid r \neq 0)} \]
Bayesian correlation test

\[
BF_{01} = \frac{p(x, y \mid r = 0)}{p(x, y \mid r \neq 0)} = \frac{\int p(x, y \mid r = 0, \theta) p(\theta) \, d\theta}{\int p(x, y \mid r \neq 0, \theta) p(\theta) \, d\theta}
\]

Parameters of the assumed model

Prior over parameter values

Hence, we want to compute

\[
BF_{01} = \frac{p(D \mid H_0)}{p(D \mid H_1)} = \frac{p(x, y \mid r = 0)}{p(x, y \mid r \neq 0)}
\]

Relative Support for President

\(r = .39\)
Bayesian correlation test

\[
BF_{01} = \frac{p(x, y \mid r = 0)}{p(x, y \mid r \neq 0)} = \frac{\int p(x, y \mid r = 0, \theta) p(\theta) d\theta}{\int p(x, y \mid r \neq 0, \theta) p(\theta) d\theta}
\]

Hence, we want to compute

\[
BF_{01} = \frac{p(D \mid H_0)}{p(D \mid H_1)} = \frac{p(x, y \mid r = 0)}{p(x, y \mid r \neq 0)}
\]

Need to specify what we mean here
Bayesian correlation test

\[ BF_{01} = \frac{p(x, y \mid r = 0)}{p(x, y \mid r \neq 0)} = \frac{\int p(x, y \mid r = 0, \theta) p(\theta) d\theta}{\int p(x, y \mid r \neq 0, \theta) p(\theta) d\theta} \]

\[ BF_{01} = \frac{p(x, y \mid r = 0)}{p(x, y \mid r \neq 0)} = \frac{\int p(x, y \mid r = 0, \theta) p(\theta) d\theta}{\int p(x, y \mid r, \theta) p(r) p(\theta) d\theta dr} \]

Hence, we want to compute

\[ BF_{01} = \frac{p(D \mid H_0)}{p(D \mid H_1)} = \frac{p(x, y \mid r = 0)}{p(x, y \mid r \neq 0)} \]
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\]

How to proceed from here?

**Naive approach**

1. Plug in bivariate normal distribution
2. Specify prior over \( r \)
3. Specify prior over \( \theta = \{\mu_1, \mu_2, \sigma_1, \sigma_2\} \)
Bayesian correlation test

\[
BF_{01} = \frac{p(x, y \mid r = 0)}{p(x, y \mid r \neq 0)} = \int p(x, y \mid r = 0, \theta) p(\theta) d\theta = \int p(x, y \mid r, \theta) p(r) p(\theta) d\theta dr
\]

How to proceed from here?

Smarter approach: ask the internet
A default Bayesian hypothesis test for correlations and partial correlations

Ruud Wetzels · Eric-Jan Wagenmakers

In order to calculate the Bayes factor for the JZS (partial) correlation test, we conceptualize these Bayesian tests as a comparison between two regression models, such that the test becomes equivalent to a variable selection test for linear regression (i.e., a test of whether or not the regression coefficient $\beta$ should be included in the model). This conceptualization allows us to exploit the JZS prior distribution. Computer code for calculating the JZS Bayes factors is presented in the Appendix.
Bayesian correlation test

\[
BF_{01} = \frac{p(x, y \mid r = 0)}{p(x, y \mid r \neq 0)} = \int \frac{p(x, y \mid r = 0, \theta) p(\theta) d\theta}{\int p(x, y \mid r, \theta) p(r) p(\theta) d\theta dr}
\]

How to proceed from here?

Wetzels & Wagenmaker’s approach:
1. Assume a JZS prior on \( r \) [an “uninformative” prior]
2. Now the BF can be computed analytically and depends only on \( r_{\text{sample}} \) and \( n \).
Bayesian stats in action
JASP:

- Free
- Similar interface as SPSS
- Bayesian and frequentist tests
- Powered by BayesFactor for R

BayesFactor for R

- Free
- Gives much more control over what you’re doing than JASP
Bayesian correlation test results

Frequentist approach:
• $p = 0.007$
• CI = [.12; .62]

Bayesian approach:
• $BF_{10} = 6.33$
• CI = [.11; .60]

JASP result:
Bayesian correlation test results

Test #2: prior belief is that $r$ is **positive**

**Frequentist approach:**
- $p = 0.003$
- CI $= [0.16; 1.0]$

**(CONFIDENCE interval)**

**Bayesian approach:**
- $BF_{+0} = 12.61$
- CI $= [0.11; 0.60]$

**(CREDIBLE interval)**

![Graph showing Bayesian correlation test results with BF$_{+0}$ and CI intervals.](image)
Bayesian correlation test results

Test #3: prior belief is that \( r \) is **negative**

Frequentist approach:
- \( p = 0.997 \)
- \( CI = [-1, 0.58] \)

Bayesian approach:
- \( BF_{-0} = 0.052 \)
- \( CI = [-0.14, -0.001] \)

(CONFIDENCE interval) (CREDIBLE interval)
Example #2:

t-test
T-test: frequentist approach

$H_0: \delta = 0$

No difference in salary between men and women

**Frequentist approach:**
1. Compute t-statistic
2. Compute p value (based on $t$ and $n$)

**Result:** $p = 0.21$

**Interpretation:**
“Assuming $H_0$ is true, we would find a test statistics as extreme (or more extreme) as in our sample in 21% of samples drawn from this population”

**Conclusion**
None – high $p$ value does not imply $H_0$ to be true
T-test: Bayesian approach

Annual salary

Male  Female

|$24,000
|$26,000
|$28,000
|$30,000
|$32,000
|$34,000
|$36,000
|$38,000
|$40,000
|$42,000

H0: δ = 0
H1: δ ≠ 0

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Bayesian t tests for accepting and rejecting the null hypothesis

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AND
GEOFFREY IVERSON
University of California, Irvine, California

Progress in science often comes from discovering invariances in relationships among variables; these invariances often correspond to null hypotheses. As is commonly known, it is not possible to state evidence for the null hypothesis in conventional significance testing. Here we highlight a Bayes factor alternative to the conventional t test that will allow researchers to express preference for either the null hypothesis or the alternative. The Bayes factor has a natural and straightforward interpretation, is based on reasonable assumptions, and has better properties than other methods of inference that have been advocated in the psychological literature. To facilitate use of the Bayes factor, we provide an easy-to-use, Web-based program that performs the necessary calculatuions.

Advances in science often come from identifying invariances—those elements that stay constant when others change. Kepler, for example, described the motion of planets. From an Earth-bound vantage point, planets seem to have strange and variable orbits. Not only do they differ in their speeds and locations, they even appear to back-
T-test: Bayesian approach

H0: \( \delta = 0 \)
H1: \( \delta \neq 0 \)

\[
BF_{01} = \frac{p(D \mid H_0)}{p(D \mid H_1)} = \frac{p(D \mid \delta = 0)}{p(D \mid \delta \neq 0)}
\]

Approach
- Assume Cauchy prior on effect size
- Assume Jeffreys prior on variance, \( p(\sigma^2) \propto 1/\sigma^2 \)
- Compute BF as follows:

\[
B_{01} = \frac{\left(1 + \frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}}{\int_0^\infty (1 + Ng)^{-1/2} \left(1 + \frac{t^2}{(1 + Ng)\nu}\right)^{-\frac{\nu+1}{2}} (2\pi)^{-1/2} g^{-3/2} e^{-1/(2g)} dg}
\]

\( t = t \) statistic, \( N = \# \) measurements, \( \nu = \# \) DoF = \( N - 1 \)
T-test: Bayesian approach

Cauchy prior
(like a normal, but sharper and fatter tails)

Default width (b=0.707)

Max width in JASP (b=2.0)
Bayes Factor Robustness Check

- max $BF_{10}$: 0.9999 at $r = 5e-04$
- user prior: $BF_{10} = 0.4604$
- wide prior: $BF_{10} = 0.3525$
- ultrawide prior: $BF_{10} = 0.2615$

Evidence for $H_1$

Evidence for $H_0$

Anecdotal
Moderate
Strong
Example #3:

ANOVA & Regression
Default Bayes factors for ANOVA designs

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ABSTRACT

Bayes factors have been advocated as superior to p-values for assessing statistical evidence in data. Despite the advantages of Bayes factors and the drawbacks of p-values, inference by p-values is still nearly ubiquitous. One impediment to the adoption of Bayes factors is a lack of practical development, particularly a lack of ready-to-use formulas and algorithms. In this paper, we discuss and expand a set of default Bayes factor tests for ANOVA designs. These tests are based on multivariate generalizations of Cauchy priors on standardized effects, and have the desirable properties of being invariant with respect to linear transformations of measurement units. Moreover, these Bayes factors are computationally convenient, and straightforward sampling algorithms are provided. We cover models with fixed, random, and mixed effects, including random interactions, and do so for within-subject, between-subject, and mixed designs. We extend the discussion to regression models with continuous covariates. We also discuss how these Bayes factors may be applied in nonlinear settings, and show how they are useful in differentiating between the power law and the exponential law of skill acquisition. In sum, the current development makes the computation of Bayes factors straightforward for the vast majority of designs in experimental psychology.

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Default Bayes Factors for Model Selection in Regression

Jeffrey N. Rouder
University of Missouri

Richard D. Morey
University of Groningen
Bayesian approach to simple linear regression


Assumed model
\[ y = \alpha + \beta x + \varepsilon \]

\( \alpha \) = intercept
\( \beta \) = slope
\( \varepsilon \) = random error (Gaussian)

Frequentist vs Bayesian approach
• Same assumed underlying model
• Same questions/hypotheses
• Different way of quantifying evidence
Bayesian approach to simple linear regression

\[
\frac{p(H_0 | D)}{p(H_1 | D)} = \frac{p(D | H_0)}{p(D | H_1)} \times \frac{p(H_0)}{p(H_1)}
\]

Posterior ratio \quad Bayes factor \quad Prior ratio

Assumed model
\[
y = \alpha + \beta x + \epsilon
\]

The hypotheses are:
\[
H_0: \beta = 0 \\
H_1: \beta \neq 0
\]

\[
BF_{01} = \frac{p(D | H_0)}{p(D | H_1)} = \frac{p(D | \beta = 0)}{p(D | \beta \neq 0)}
\]

Computable

Uncomputable unless we specify what we mean with “\(\beta \neq 0\)”

-> Cauchy prior
Bayesian approach to simple linear regression

$$\text{BF}_{01} = \frac{p(D \mid H_0)}{p(D \mid H_1)} = \frac{p(D \mid \beta = 0)}{p(D \mid \beta \neq 0)}$$

Computable

Uncomputable unless we specify what we mean with “\(\beta \neq 0\)”

\(\Rightarrow\) Cauchy prior

Cauchy prior
(like a normal, but sharper and fatter tails)
Bayesian approach to simple linear regression

Assumed model
\[ y = \alpha + \beta x + \varepsilon \]

\( \alpha \) = intercept
\( \beta \) = slope
\( \varepsilon \) = random error (Gaussian)


| Models          | P(M) | P(M|data) | BFM  | BF10  | error % |
|-----------------|------|----------|------|-------|---------|
| Null model      | 0.500| 0.046    | 0.048| 1.000 | 0.003   |
| LSD dose        | 0.500| 0.954    | 20.852| 20.852| 0.003   |
Bayesian approach to simple linear regression

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Model Comparison - Math score

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| Null model     | 0.500| 0.046    | 0.048| 1.000|         |
| LSD dose       | 0.500| 0.954    | 20.852| 20.852| 0.003   |

Prior model evidence
Bayesian approach to simple linear regression

Assumed model
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\( \alpha = \) intercept
\( \beta = \) slope
\( \varepsilon = \) random error (Gaussian)


| Models          | P(M) | P(M|data) | BF_M | BF_{10} | error % |
|-----------------|------|----------|------|----------|---------|
| Null model      | 0.500| 0.046    | 0.048| 1.000    | 0.003   |
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| LSD dose     | 0.500| 0.954   | 20.852| 20.852| 0.003   |

Change from prior to posterior odds (=Bayes factor of model Mx relative to all others)
Bayesian approach to simple linear regression

Assumed model

\[ y = \alpha + \beta x + \varepsilon \]

\( \alpha = \) intercept
\( \beta = \) slope
\( \varepsilon = \) random error (Gaussian)


Model Comparison - Math score

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Model Comparison - Math score

| Models       | P(M) | P(M|data) | BF_M | BF_10 | error % |
|--------------|------|----------|------|-------|---------|
| Null model   | 0.500| 0.046    | 0.048| 1.000 |         |
| LSD dose     | 0.500| 0.954    | 20.852| 20.852| 0.003   |

BF estimation error
Example with multiple regressors
(aka covariates)
Example with multiple regressors


<table>
<thead>
<tr>
<th>Origin</th>
<th>Avg weekly wage ($)</th>
<th>English speaking (%)</th>
<th>Literate (%)</th>
<th>&gt;5 years in US (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Armenian</td>
<td>9.73</td>
<td>54.9</td>
<td>92.1</td>
<td>54.6</td>
</tr>
<tr>
<td>Bohemian/Moravian</td>
<td>13.07</td>
<td>66.0</td>
<td>96.8</td>
<td>71.2</td>
</tr>
<tr>
<td>Bulgarian</td>
<td>10.31</td>
<td>20.3</td>
<td>78.2</td>
<td>8.5</td>
</tr>
<tr>
<td>Canadian (French)</td>
<td>10.62</td>
<td>79.4</td>
<td>84.1</td>
<td>86.7</td>
</tr>
<tr>
<td>Canadian (Other)</td>
<td>14.15</td>
<td>100.0</td>
<td>99.0</td>
<td>90.8</td>
</tr>
<tr>
<td>Croatian</td>
<td>11.37</td>
<td>50.9</td>
<td>70.7</td>
<td>38.9</td>
</tr>
<tr>
<td>Danish</td>
<td>14.32</td>
<td>96.5</td>
<td>99.2</td>
<td>85.4</td>
</tr>
<tr>
<td>Dutch</td>
<td>12.04</td>
<td>86.1</td>
<td>97.9</td>
<td>81.9</td>
</tr>
<tr>
<td>English</td>
<td>14.13</td>
<td>100.0</td>
<td>98.9</td>
<td>80.6</td>
</tr>
<tr>
<td>Finnish</td>
<td>13.27</td>
<td>50.3</td>
<td>99.1</td>
<td>53.6</td>
</tr>
<tr>
<td>Flemish</td>
<td>11.07</td>
<td>45.6</td>
<td>92.1</td>
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</tr>
<tr>
<td>French</td>
<td>12.92</td>
<td>68.6</td>
<td>94.3</td>
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<tr>
<td>German</td>
<td>13.63</td>
<td>87.5</td>
<td>98.0</td>
<td>86.4</td>
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<tr>
<td>Greek</td>
<td>8.41</td>
<td>33.5</td>
<td>84.2</td>
<td>18.0</td>
</tr>
<tr>
<td>Hebrew (Russian)</td>
<td>12.71</td>
<td>74.7</td>
<td>93.3</td>
<td>57.1</td>
</tr>
<tr>
<td>Hebrew (Other)</td>
<td>14.37</td>
<td>79.5</td>
<td>92.8</td>
<td>73.8</td>
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<tr>
<td>Irish</td>
<td>13.01</td>
<td>100.0</td>
<td>96.0</td>
<td>90.6</td>
</tr>
<tr>
<td>Italian (Northern)</td>
<td>11.28</td>
<td>58.8</td>
<td>85.0</td>
<td>55.2</td>
</tr>
<tr>
<td>Italian (Southern)</td>
<td>9.61</td>
<td>48.7</td>
<td>69.3</td>
<td>47.8</td>
</tr>
</tbody>
</table>

Assumed model: \( y = \alpha + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \varepsilon \)
Example with multiple regressors

Dependent variable: average weekly salary
Covariates: (1) english speaking (%), (2) literate (%), (3) >5 years in US (%)

**FREQUENTIST RESULT**

<table>
<thead>
<tr>
<th>Coefficients</th>
<th>Unstandardized</th>
<th>Standard Error</th>
<th>Standardized</th>
<th>t</th>
<th>p</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>intercept</td>
<td>2.576</td>
<td>1.312</td>
<td></td>
<td>1.964</td>
<td>0.059</td>
</tr>
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<td>English speaking (%)</td>
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<td>0.093</td>
</tr>
<tr>
<td>Literate (%)</td>
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<td>0.020</td>
<td>0.497</td>
<td>3.930</td>
<td>&lt;.001</td>
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<tr>
<td>&gt;5 years in US (%)</td>
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<td>0.021</td>
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**BAYESIAN RESULT**

<table>
<thead>
<tr>
<th>Model Comparison - Avg weekly wage ($)</th>
</tr>
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<tbody>
<tr>
<td>Models</td>
</tr>
<tr>
<td>Null model</td>
</tr>
<tr>
<td>English speaking (%)</td>
</tr>
<tr>
<td>Literate (%)</td>
</tr>
<tr>
<td>English speaking (%) + Literate (%)</td>
</tr>
<tr>
<td>&gt;5 years in US (%)</td>
</tr>
<tr>
<td>English speaking (%) + &gt;5 years in US (%)</td>
</tr>
<tr>
<td>Literate (%) + &gt;5 years in US (%)</td>
</tr>
<tr>
<td>English speaking (%) + Literate (%) + &gt;5 years in US (%)</td>
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Example with multiple regressors

Dependent variable: average weekly salary
Covariates: (1) english speaking (%), (2) literate (%), (3) >5 years in US (%)
Example with multiple regressors

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</tr>
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</table>

**BAYESIAN RESULT**

| Effects                  | P(incl) | P(incl|data) | BFInclusion |
|--------------------------|---------|---------|-------------|
| English speaking (%)     | 0.500   | 0.804   | 4.094       |
| Literate (%)             | 0.500   | 0.990   | 102.066     |
| >5 years in US (%)       | 0.500   | 0.282   | 0.392       |
Take-home points

#1

‘NHST’ is a widespread but flawed approach

(*) NHST=Null Hypothesis Significance Testing
Take-home points

#2

Evidence is best treated as a relative concept

- The Bayes Factor is by definition a relative measure
- The p-value is an absolute measure
Take-home points

#3

Ideally we want to be able to both reject and accept hypotheses

- The Bayes Factor can quantify evidence in both directions
- The p-value can only reject
- Disregard of “null results” is a main driver behind the replication crisis
Ideally we want statistical evidence to be conditioned only on data

- The Bayes Factor has this property
- The p-value depends on data collection stopping rule!
Take-home points

#5

The Bayesian approach requires specifying priors

- Some see this as a curse
- Others see this as an opportunity to include prior knowledge
Take-home points

#6

Bayesians quantify belief, frequentists compute long-run frequencies
Take-home points

#7

Above all: make sure you know what you are doing!

Mindful Bayesian
> Mindful frequentist
>>>>>>>
Mindless Bayesian
> Mindless Frequentist
RECOMMENDED READING
Mindless statistics
Gerd Gigerenzer*

Max Planck Institute for Human Development, Lentzeallee 94, 14195 Berlin, Germany

Abstract

Statistical rituals largely eliminate statistical thinking in the social sciences. Rituals are indispensable for identification with social groups, but they should be the subject rather than the procedure of science. What I call the “null ritual” consists of three steps: (1) set up a statistical null hypothesis, but do not specify your own hypothesis nor any alternative hypothesis, (2) use the 5% significance level for rejecting the null and accepting your hypothesis, and (3) always perform this procedure. I report evidence of the resulting collective confusion and fears about sanctions on the part of students and teachers, researchers and editors, as well as textbook writers.

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Keywords: Rituals; Collective illusions; Statistical significance; Editors; Textbooks

...no scientific worker has a fixed level of significance at which from year to year, and in all circumstances, he rejects hypotheses; he rather gives his mind to each particular case in the light of his evidence and his ideas.

Sir Ronald A. Fisher (1956)

I once visited a distinguished statistical textbook author, whose book went through many editions, and whose name does not matter. His textbook represents the relative best in the social sciences. He was not a statistician; otherwise, his text would likely not have been used in a psychology class or political science class. The student learns Bayesian statistics, the importance of sample size, and the benefit of statistical power. He is almost certain to conclude the existence of a relationship, and almost certain to conclude that the relationship is in the direction of his hypothesis. He is least likely to conclude the possibility of the null hypothesis. No textbook uses any statistical model. There are no examples. No references. No exercises. No problems. No solutions. It is the best textbook on statistical methodology, and will perhaps be the textbook that will dominate the social sciences for years to come. And it is in the true spirit of mindless statistics.

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Editorial Commentary

Surrogate Science: The Idol of a Universal Method for Scientific Inference

Gerd Gigerenzer
Max Planck Institute for Human Development

Julian N. Marewski
University of Lausanne

The application of statistics to science is not a neutral act. Statistical tools have shaped and were also shaped by their objects. In the social sciences, statistical methods fundamentally changed research practice, making statistical inference its centerpiece. At the same time, textbook writers in the social sciences have transformed rivaling statistical systems into an apparently monolithic method that could be used mechanically. The idol of a universal method for scientific inference has been worshipped since the “inference revolution” of the 1920s. Because no such method has ever been found, surrogates have been created, most notably the quest for significant p-values. This form of surrogate science fosters delusions and borderline cheating and has done much harm, creating, for one, a flood of irreproducible results. Proponents of the “Bayesian revolution” should be wary of chasing yet another chimera: an apparently universal inference procedure. A better path would be to promote both an understanding of the various devices in the “statistical toolbox” and informed judgment to select among these.

Keywords: research methods; regression analysis; psychometrics; Bayesian methods

No scientific worker has a fixed level of significance at which from year to year, and in all circumstances, he rejects hypotheses; he rather gives his mind to each particular case in the light of his experience and knowledge.

Acknowledgements: I would like to thank Ira Pasternak for helping with the case of misappropriation values in the Academy of Sciences, and Peter Dienes for helping with the concept of adaptive significance.

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RECOMMENDED READING

421
Bayesian Inference for Psychology. Part I: Theoretical Advantages and Practical Ramifications

Eric-Jan Wagenmakers¹, Maarten Marsman¹, Tahira Jamil¹, Alexander Ly¹, Josine Verhagen¹, Jonathon Love¹, Ravi Selker¹, Quentin F. Gronau¹, Martin Smíra², Sacha Epskamp¹, Dora Matzke¹, Jeffrey N. Rouder³, & Richard D. Morey⁴
¹ University of Amsterdam
² Masaryk University
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Abstract
Bayesian parameter estimation and Bayesian hypothesis testing present attractive alternatives to classical inference using confidence intervals and $p$ values. In part I of this two-part series we outline ten prominent advantages of the Bayesian approach. Many of these advantages translate to concrete opportunities for pragmatic researchers. For instance, Bayesian hypothesis testing allows researchers to quantify evidence and monitor its progression as data come in, without needing to know the intention with which the data were collected. We end by countering several objections to Bayesian hypothesis testing. Part II of this series discusses JASP, a free and open source software program that makes it easy to conduct Bayesian estimation and testing for a range of popular statistical scenarios (Love et al., this issue).

Keywords: distribution, theoretical reading
Bayesian Inference for Psychology. Part II: Example Applications with JASP

Eric-Jan Wagenmakers\textsuperscript{1}, Jonathon Love\textsuperscript{1}, Maarten Marsman\textsuperscript{1}, Tahira Jamil\textsuperscript{1}, Alexander Ly\textsuperscript{1}, Josine Verhagen\textsuperscript{1}, Ravi Selker\textsuperscript{1}, Quentin F. Gronau\textsuperscript{1}, Damian Dropmann\textsuperscript{1}, Bruno Boutin\textsuperscript{1}, Frans Me chiropr\textsuperscript{1}, Patrick Knight\textsuperscript{1}, Akash Raj\textsuperscript{2}, Erik-Jan van Kesteren\textsuperscript{1}, Johnny van Doorn\textsuperscript{1}, Martin Smíra\textsuperscript{3}, Sacha Epskamp\textsuperscript{1}, Alexander Etz\textsuperscript{4}, Dora Matzke\textsuperscript{1}, Jeffrey N. Rouder\textsuperscript{5}, Richard D. Morey\textsuperscript{6}

\textsuperscript{1} University of Amsterdam
\textsuperscript{2} Bili Institute of Technology and Science
\textsuperscript{3} Masaryk University
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Abstract

Bayesian hypothesis testing presents an attractive alternative to \( p \) value hypothesis testing. Part I of this series outlined several advantages of Bayesian hypothesis testing, including the ability to quantify evidence and the ability to monitor and update this evidence as data come in, without the need to know the intention with which the data were collected. Despite these advantages, Bayesian hypothesis tests are still reported relatively rarely. An important impediment to the widespread adoption of Bayesian tests is arguably the lack of user-friendly software for the run-of-the-mill statistical problems that confront psychologists for the analysis of almost every experiment: the \( t \)-test, ANOVA, correlation, regression, and contingency tables. In Part II of this series, we introduce JASP (Just Another Statistical Program), a free, open-source, graphical user interface for statistical analysis with a syntax similar to familiar \( R \) packages. JASP is designed to facilitate Bayesian hypothesis testing for psychologists and other users who are not familiar with the \( R \) programming language, and it provides a user-friendly and intuitive interface for conducting Bayesian analyses.

RECOMMENDED READING

Some extra slides
# Fisher vs Neyman-Pearson

<table>
<thead>
<tr>
<th>Fisher's approach</th>
<th>Neyman-Pearson's approach</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Outcome:</strong> significant / non-significant</td>
<td><strong>Outcome:</strong> accept / reject</td>
</tr>
<tr>
<td>( p ) is a measure of evidence against H0</td>
<td>( p ) is NOT a measure of evidence and should not be interpreted</td>
</tr>
<tr>
<td>An alternative hypothesis <strong>cannot</strong> be specified</td>
<td>An alternative hypothesis <strong>must</strong> be specified</td>
</tr>
<tr>
<td>Does not have a concept of &quot;power&quot;</td>
<td>Power has to be specified prior to the experiment</td>
</tr>
<tr>
<td>A single rejection of H0 is the start, not the end, of an investigation. Replication needed and meta-analyses are useful</td>
<td>A single rejection is meaningless – the framework only guarantees long-term type-1 and type-2 error rates but does not allow to make inference about a single case.</td>
</tr>
</tbody>
</table>

Presently, much statistical testing in psychology research is an "inconsistent hybrid that every decent statistician would reject" (Gigerenzer, 2004)
Why should we bother about statistical literacy?

Main findings
1) Only 36% of significant results replicated
2) Effect sizes shrunk by ~50% in the replications

What caused the crisis?

A toxic mix of the following:

- Publication pressure
- Disregard for “null findings”

... which incentivizes poor methodological hygiene:

- Hide null findings (file drawer problem)
- Test many variables, report few (fishing)
- Try many tests, report few (p-hacking)
- Post-hoc hypothesizing (HARK-ing)
- ...

Bayesian stats is not a miracle cure, but understanding the Bayesian approach will make you a more insightful consumer of statistics – which will likely lead to better statistical practices even if you stick to the frequentist methods.