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# Global Psychophysical Judgments of Intensity: Summary of a Theory and Experiments

R. Duncan Luce
University of California, Irvine

Ragnar Steingrimsson
New York University

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This chapter has three thrusts: (1) It formulates in a common framework mathematical representations of two global sensory procedures: summation of intensity and the method of ratio production (Luce, 2002, 2004). Until recently, these two topics have not been treated together in the literature. (2) Although the psychophysical representations we arrive at include both free parameters and free functions, a message of this work, especially illustrated in Luce and Steingrimsson (2005a,b)[1], is that one can evaluate their adequacy without ever estimating either the parameters or the functions. Rather, it is sufficient just to evaluate parameter-free behavioral properties that give rise to the representations. (3) A closely related message is that, to the degree that the theory holds, no individual differences arise in the defining behavioral properties except, of course, for the fact that each person has his or her own sense of the relative intensities of two stimuli, i.e., the subjective intensity ordering. At the same time, the potential exists for substantial individual differences in the representations in the following sense: there is a strictly increasing psychophysical function and a strictly increasing numerical distortion function that are not otherwise prescribed without additional axioms. The work on the forms of these functions, although quite well developed, is not yet in final manuscript form. Nonetheless, we cover it in some detail in Sections 5 and 6.

The chapter describes the theory and discusses our joint experimental program to test it. Some of this, SL-I and SL-II, is in articles now under revision for publication.

[1] In the remainder of the chapter, the four collaborative articles by Steingrimsson and Luce are identified as SL-I, SL-II, etc.

Additional manuscripts, SL-III and SL-IV, on the forms of the psychophysical function and the forms of the weighting function are nearing completion. Portions of all of these, including much of the experimental work, derive in part from Steingrimsson's (2002) UCI dissertation.

We formulate the exposition in terms of loudness judgments about pure tones of the same frequency and phase. However, many other interpretations of the primitives are possible and each must be evaluated empirically in a separate experimental program. An investigation of brightness summation across the two eyes is currently underway by the second author.

# 1 Primitives and Representations

## 1.1 Ordering of joint presentations

Let $x$ denote the signal intensity less the threshold intensity of a pure tone presented to the left ear. We stress that we mean an intensity difference, not the more usual intensity ratio that leads to differences in dB. Let $u$ denote the intensity less the threshold of a pure tone of the same frequency and phase presented to the right ear. Thus, $0$ is the threshold intensity in each ear. The notation $(x, u)$ denotes the simultaneous presentation of $x$ and $u$ in the left and the right ear, respectively. The assumption of a fixed threshold is, of course, an idealization–in reality, the threshold is a random variable which we idealize as a single number. In making this idealization and others like it, we rely on the position articulated shortly before his death by the youthful philosopher Frank Ramsey (1931, p. 168; 1964, p. 70) in talking about decision making under uncertainty: "Even in physics we cannot maintain that things that are equal to the same thing are equal to one another unless we take 'equal' not as meaning 'sensibly equal' but a fictitious or hypothetical relation. I do not want to discuss the metaphysics or epistemology of this process, but merely to remark that if it is allowable in physics it is allowable in psychology also. The logical simplicity characteristic of the relations dealt with in a science is never attained by nature alone without any admixture of fiction."

In the task we used, respondents were asked to judge if $(x, u)$ is at least as loud as $(y, v)$, which is denoted $(x, u) \succsim (y, v)$. The results we report show conditions such that this loudness ordering is reflected by a numerical mapping $\Psi : \mathbb{R}_+ \times \mathbb{R}_+ \xrightarrow{onto} \mathbb{R}_+$, where $\mathbb{R}_+ := [0, \infty[$, [2] that is strictly increasing in each variable and is order preserving, i.e.,

$$(x, u) \succsim (y, v) \quad \Leftrightarrow \quad \Psi(x, u) \geq \Psi(y, v), \tag{1}$$

$$\Psi(0, 0) \quad = \quad 0. \tag{2}$$

And we assume that loudness and intensity agree to the extent that

$$\begin{aligned} (x, 0) \quad &\succsim \quad (y, 0) \Leftrightarrow x \geq y, \\ (0, u) \quad &\succsim \quad (0, v) \Leftrightarrow u \geq v. \end{aligned}$$

[2] The notation $A := B$ means that $A$ is defined to be $B$.

Thus, $\Psi(x,0)$ and $\Psi(0,u)$ are each strictly increasing.

We assume that the respondent can always establish matches of three types to each stimulus:

$$(x,u) \sim (z_l,0),\ (x,u) \sim (0,z_r),\ (x,u) \sim (z_s,z_s), \tag{3}$$

where $\sim$ means equally loud. The left and right matches $z_l$ and $z_r$ are called *asymmetric* and $z_s$ is called a *symmetric* match. Symmetric matches have the decided advantage of reducing the degree of localization change between $(x,u)$ and the matching pair. The asymmetric matches encounter some difficulties which we discuss in Sec. 5.1 and overcome in Sec. 5.2.

Note that each of the $z$'s is a function of both $x$ and $u$. To make that explicit and suggestive we use an operator notation:

$$x \oplus_i u := z_i \quad (i = l, r, s). \tag{4}$$

It is not difficult to show that each of the $\oplus_i$ defined by (4) is, indeed, a binary operation that is defined for each pair $(x,u)$ of intensities. The operator $\oplus_i$ may be referred to as a *summation operator*; however, one must not confuse $\oplus_i$with $+$. Some readers of our work have expressed discomfort over the fact that we can explore, for example, whether or not the operation is associative, i.e.,

$$x \oplus_i (y \oplus_i z) = (x \oplus_i y) \oplus_i z \quad (i = l, r, s) \tag{5}$$

despite the fact that the notation

$$(x,(y,z)) \sim ((x,y),z)$$

is, itself, meaningless. Such a defined operator is, however, a familiar and commonly used trick in the theory of measurement to map something with two or more dimensions into a structure on a single dimension. See for example the treatment of conjoint measurement in Sec. 6.2.4 of Krantz, Luce, Suppes, and Tversky (1971) and in Sec. 19.6 of Luce, Krantz, Suppes, and Tversky (1990).

One can show (see Proposition 1 of Luce, 2002) that $\succsim$ is a weak order (i.e., transitive and connected), that $(x,u)$ is strictly increasing in each variable, and that $0$ is a *right identity* of $\oplus_l$, i.e.,

$$x = x \oplus_l 0, \tag{6}$$

and $0$ is a *left identity* of $\oplus_r$, i.e.,

$$u = 0 \oplus_r u, \tag{7}$$

whereas $0$ is not an identity of $\oplus_s$ at all. However, the symmetric operation is *idempotent* in the sense that

$$x \oplus_s x = x. \tag{8}$$

These properties play important roles in some of the proofs.

We assume that the function $\Psi(x,u)$ is decomposable in the sense that it depends just on its components $\Psi(x,0)$ and $\Psi(0,u)$,

$$\Psi(x,u) = F[\Psi(x,0), \Psi(0,u)]. \tag{9}$$

One natural question is: What is the nature of that dependence, i.e., what is the form of $F$? A second natural question is: How do $\Psi(x,0)$ and $\Psi(0,u)$ depend on the physical intensities $x$ and $u$, respectively? These are ancient problems with very large literatures which we make no attempt to summarize here. Some references will appear below. Neither question, it should be mentioned, is resolved in any fully satisfactory manner if one restricts attention just to the primitive ordering $\succsim$ of the conjoint structure of intensities, $\langle \mathbb{R}_+ \times \mathbb{R}_+, \succsim \rangle$. Some additional structure beyond the ordering is needed to achieve an effective theory. Below in Sec. 4 we will encounter two examples of such additional linking structures, which in these cases are two forms of a distribution law. This important point, which is familiar from physics, does not seem to have been as widely recognized by psychologists as we think that it should be.

Two points should be stressed. The first is that the theory is not domain specific, which means that it has many potential interpretations in addition to our auditory one. For example, also in audition, Karin Zimmer and Wolfgang Ellermeier[3], interpreted $(x,u)$ to mean a brief signal of intensity $x$ followed almost immediately by a another brief signal of intensity $u$. Other interpretations, using visual stimuli, are brightness summation of hemifields or across the two eyes are some other possibilities. Each conceivable interpretation will, of course, require separate experimental verification, although drawing on our experience with the two ear experiments should be beneficial.

The second point is that the approach taken here is entirely behavioral and so is independent of any particular biological account of the behavior. Consequently, we do not make any attempt to draw any such conclusions from our results.

## 1.2 Ratio productions

To the ordering of signal pairs we add the independent structure of a generalized form of *ratio production*[4]. This entails the presentation to respondents of a positive number $p$ and the stimuli $(x,x)$ and $(y,y)$, where $y < x$, with the request to produce the stimulus $(z,z)$ for which the loudness "interval" from $(y,y)$ to $(z,z)$ is perceived to stand in the ratio $p$ to the "interval" from $(y,y)$ to $(x,x)$. Because the $z$ chosen by respondent is a function of $p$, $x$, and $y$, we may again represent that functional dependence in operational form as:

$$(x,x) \circ_p (y,y) := (z,z). \tag{10}$$

This operation, which we call (subjective) *ratio production* is somewhat like Stevens' magnitude production (for a summary, see Stevens, 1975) which is usually described as finding the signal $(z,z)$ that stands in proportion $p$ to stimulus $(x,x)$. Thus, his method is the special case of ours but with $(y,y) = (0,0)$–the threshold intensity or below. Thus, $(x,x) \circ_p (0,0) = (z,z)$.

We assume two things about $\circ_p$: (i) it is strictly increasing in the first variable and non-constant and continuous in the second, and (ii) that $\Psi$ over $\circ_p$ is also decomposable

[3]As reported at the 2001 meeting of the European Mathematical Psychology Group in Lisbon, Portugal.

[4]In a generalized *ratio estimation* the respondent is presented with two pairs of stimuli, $(y,y)$ to $(z,z)$ and $(y,y)$ to $(x,x)$ where $y < x, z$, and is asked to state the ratio $p = p(x,y,z)$ of the interval between the first two to the interval between the second two. This is discussed in SL-III and is summarized below in Sec. 6.1. This procedure is, of course, conceptually related to S. S. Stevens' magnitude estimation where no standard is provided (see after (10)).

in the sense that

$$\Psi[(x,x) \circ_p (y,y)] = G_p[\Psi(x,x), \Psi(y,y)]. \tag{11}$$

## 1.3 The representations

On the above assumptions, Luce (2002, 2004) presented necessary and sufficient qualitative conditions for the representations. These are discussed below in Secs. 3 and 4.[5]

$$\begin{aligned} \Psi(x,u) &= \Psi(x,0) + \Psi(0,u) + \delta\Psi(x,0)\Psi(0,u), && (12) \\ W(p) &= \frac{\Psi\left((x,x) \circ_p (y,y)\right) - \Psi(y,y)}{\Psi(x,x) - \Psi(y,y)} \quad (x > y \geq 0), && (13) \end{aligned}$$

where $\delta$ is a non-negative constant and $W : [0,\infty[ \stackrel{onto}{\longrightarrow} [0,\infty[$ is strictly increasing. The "summation" formula (12) has been dubbed ***p-additive*** because it is the unique polynomial function of $\Psi(x,0)$ and $\Psi(0,u)$ with $\Psi(0,0) = 0$ that can be transformed into additive form (see Sec. 3.2).

Under certain assumptions one can also show that, for some $\gamma > 0$,

$$\Psi(x,0) = \gamma\Psi(0,x), \tag{14}$$

which we call *constant bias*; however, for other assumptions constant bias is not forced. More specifically, if the properties in Secs. 3 and 4 hold for asymmetric matches, then constant bias, (14), holds in addition to the two representations (12) and (13) (Luce, 2002, 2004). In contrast, if the properties hold using symmetric matches, then one can prove that (12) holds with $\delta = 0$, that (13) holds, but that constant bias, (14), need not hold. Because constant bias seems intuitively unlikely–the ears often do not seem to be identical–we are probably going to be best off with the symmetric theory. We discuss data on whether our young respondents satisfy the assumption of having symmetric ears above threshold. We also investigate empirically whether or not $\delta = 0$ (Sec. 3.3), which has to hold if symmetric matches satisfy the conditions. If $\delta = 0$, empirical testing of the theory is simplified considerably.

Nothing in the theory giving rise to (12) and (13) dictates the explicit mathematical forms for $\Psi(x,0)$ as a function of the physical intensity $x$, for $\Psi(0,u)$ as a function of $u$, or for $W(p)$ as a function of $p$. One attempt to work out the form of $\Psi$ based just on the summation operation is summarized in Sec. 5.4. It leads to a sum of power functions. Another, also leading to a power function form, is based on ratio productions, Sec. 6.3. The experimental data make clear that our endeavors are incomplete. Our attempts to find out something about $W$, which is also incomplete, are summarized in Sec. 6.

Where do the above representations come from and how do we test them? Various testable conditions that are necessary and sufficient for the representations are outlined

[5]In Luce (2002) all of the results are presented in terms of psychophysical functions on each signal dimension, as was the first submitted version of Luce (2004). As a reviewer, Ehtibar Dzhafarov saw that they could be neatly brought together as a psychophysical function over the signal pairs, and Luce adopted that formulation.

and the results of several experimental tests are summarized. Note that we make no attempt to fit the representations themselves directly to data. In particular, no parametric assumptions are imposed about the nature of the functions $\Psi$ and $W$. In Sec. 5.4 we will see how to test for the power function form of $\Psi$ using parameter-free properties, and in Sec. 6, again using parameter-free properties, we arrive at two forms for $W$.

# 2 Methods of testing

The many experiments discussed employ empirical interpretations of the two operations. One is $x \oplus_i u := z_i \quad (i = l, r, s)$, (4), which involves estimating a value $z_i$ that is experience equal in loudness to the joint-presentation $(x, u)$. The other is $(x, x) \circ_p (y, y) := (z, z) \quad (y < x)$, (10), which involves estimating a value $z$ that makes the loudness "interval" between $(y, y)$ and $(x, x)$ be a proportion $p$ of the interval between $(y, y)$ and $(z, z)$. The first procedure is referred to as matching and the second as ratio production.

The stimuli used were, in all cases, 1,000 Hz pure tones of 100 ms duration that included a 10 ms on and off ramp. Throughout, signals are described as dB relative to SPL.

## 2.1 Matching procedure

To describe the testing, we employ the notation $\langle A, B \rangle$ to mean the presentation of stimulus $A$ followed 450 ms after by stimulus $B$, where $A$ and $B$ are joint presentations. Then the three matches of (3) are obtained using whichever is relevant of the following three trial forms:

$$\langle (x, u), (z_l, 0) \rangle, \tag{15}$$

$$\langle (x, u), (0, z_r) \rangle, \tag{16}$$

$$\langle (x, u), (z_s, z_s) \rangle. \tag{17}$$

In practice, respondents heard a tone followed 450 ms later by another tone in the left, right, or both ears, as the case might be. Respondents used key presses either to adjust the sound pressure level of $z_i$, $i = l, r, s$ (by one of four differently sized steps), to repeat the previous trial, or to indicate satisfaction with the loudness match. Following each adjustment, the altered tone sequence was played. This process was repeated until respondents were satisfied that the second tone matched the first tone in loudness.

## 2.2 Ratio production procedures

The basic trial form is $\langle \langle A, B \rangle, \langle A, C \rangle \rangle$ where $\langle A, B \rangle$ and $\langle A, C \rangle$ represent the first and the second intensity interval respectively. The $\langle A, B \rangle$ and $\langle A, C \rangle$ were separated by 750 ms, and between $A$ and $B$ (and $A$ and $C$) the delay was 450 ms.

An estimate of $x \circ_{p,i} y = v_i$, in the case of $i = s$, was obtained using the trial type

$$\langle \langle (y, y), (x, x) \rangle, \langle (y, y), (v_s, v_s) \rangle \rangle, \tag{18}$$

where the value of $v_s$ was under the respondents' control. In practice, respondents heard two tones separated by 450 ms (the first interval) then 750 ms later, another such set of tones was heard (the second interval). The first tone in both intervals is identical and more quiet than the second tone. Respondents continued to alter the sound pressure level of $v_s$ until they experienced the second loudness interval as being a proportion $p$ of the second one.

As mentioned above, the special case of $y = 0$ is an operation akin to Stevens' magnitude production, which involves finding the signal $(z, z)$ that stands in proportion $p$ to stimulus $(x, x)$. With $i = s$ was estimated using the trial type

$$\langle (x, x), (v_s, v_s) \rangle. \tag{19}$$

In practice, respondents hear two tones, separated by 450 ms, and they adjust the second tone to be a proportion $p$ of the first tone.

Trial forms in the case of $i = r, l$ are constructed in a manner analogous to (18) and (19).

## 2.3 Statistics

The four SL articles examined parameter-free null hypotheses of the form $L = R$. Not having any a priori idea of the distribution of empirical estimates we used the non-parametric Mann-Whitney U-test at the 0.05 level. To improve our statistical certainty, we used Monte Carlo simulations based on the bootstrap technique (Efron and Tibshirani, 1993) to verify that the hypothesis that both $L$ and $R$ could be argued to come from the same underlying distribution, at the 0.05 level. This was our criterion for accepting the null hypothesis, supporting the behavioral property.

## 2.4 Additional methodological observations

During the course of doing these studies we encountered and overcame or attenuated the impact of some methodological issues (details in SL-I).

The well-known time-order error–i.e., the impact of $(x, u)$ depends on whether it occurs before or after $(y, v)$–means that it is important to use some counter-balancing or ensure that the errors are balanced on the two sides of a behavioral indifference.

Some experiments require us to use an estimate from one step as input to a second one. If a median or other average from the first step is used as the input in a second step, then whatever error it contains is necessarily carried over into that second one, but all information about the variance is lost. After some experience we concluded that the results are more satisfactory if we used each individual estimate from the first step as an input to the second one. Then the errors of the first estimate are carried into the second step and average out there, while preserving the variance information.

Variability for ratio productions tends to be higher than for matching. This fact means that some attention needs to be paid to the amount of practice respondents' receive, the evaluation of results, and obtaining an adequate number of observations.

# 3 Summations and Productions Separately

Much of the mathematical formulation of the theory was first developed for utility theory (summarized in Luce, 2000) under the assumption that the following property holds:

**Joint-presentation symmetry:**

$$(x, u) \sim (u, x). \tag{20}$$

This means that the ears are identical in dealing with intensities above their respective thresholds. We know this need not always hold (e.g., single-ear deafness, exposure of one ear to percussive rifle shots), but at first we thought that it might be approximately true for young people with no known hearing defects. Note that jp-symmetry, (20), is equivalent to $\oplus_l \equiv \oplus_r$ and $\oplus_s$ all being commutative operations, i.e.,

$$x \oplus_i y = y \oplus_i x \quad (i = l, r, s).$$

## 3.1 Evidence against symmetric hearing using symmetric matching

Using symmetric matching

$$z = x \oplus_s y \text{ and } z' = y \oplus_s x$$

were obtained using trial of form (17). There were from 34-50 matches per stimulus per respondent. We used tones with intensities $a$ = 58 dB, $b$ = 64 dB, and $c$ = 70 dB SPL, which gave rise to six ordered stimulus pairs: $(a, b)$, $(a, c)$, $(b, c)$ and $(b, a)$, $(c, a)$, $(b, c)$. For each pair, we tested statistically whether the null hypothesis $z = z'$ held. With 15 respondents there were 45 tests of which 23 were rejected. The pattern of results suggests that jp-symmetry fails for at least 12 of the 15 respondents.

The negative outcome of this experiment motivated the developments in Luce (2002, 2004) where jp-symmetry is not assumed to hold.

In Sec. 5.1 we turn to the use of asymmetric matches to study the properties underlying the representation. They sometimes exhibit an undesirable phenomenon for which we provide an explanation, and after the fact show that the properties below using asymmetric matches, under stated conditions, are unaffected.

## 3.2 Thomsen condition

The representation (12) with $\delta = 0$,

$$\Psi(x, u) = \Psi(x, 0) + \Psi(0, u), \tag{21}$$

is nothing but an additive conjoint representation (Ch. 6 of Krantz, Luce, Suppes, & Tversky, 1971). And, for $\delta > 0$, the p-additive representation, (12), can be rewritten as

$$1 + \delta\Psi(x, u) = [1 + \delta\Psi(x, 0)]\,[1 + \delta\Psi(0, u)]\,,$$

so under the transformation

$$\Theta(x,u) = \ln\left[1 + \delta\Psi(x,u)\right], \tag{22}$$

the conjoint structure again has an additive representation. So data bearing on the existence of an additive presentation is of interest whether or not $\delta = 0$.

With our background assumptions–weak ordering, strict monotonicity, solvability, and that intensity changes in either ear affect loudness–we can show a property that is analogous to the numerical Archimedean property that for any two positive numbers $a$ and $b$, one can find an integer $n$ such that $na > b$. Thus, by Krantz et al. (1971, Ch. 6) we need only the following condition in order to construct an additive representation $\Theta$.

**Thomsen condition:**

$$\left.\begin{array}{l}(x,t) \sim (z,v)\\(z,u) \sim (y,t)\end{array}\right\} \Longrightarrow (x,u) \sim (y,v) \tag{23}$$

If all of the $\sim$ are replaced by $\succsim$, the resulting condition is called *double cancellation*. The reason for that term is that the condition can be paraphrased as involving the two "cancellations" $t$ and $z$, each of which appears on each side of the hypotheses, to arrive at the conclusion.

We know of no empirical literature in audition, other than our study described below, that tests the Thomsen condition, *per se*. What has been published concerning conjoint additivity all examined double cancellation, which we feel is a somewhat less sensitive challenge than the Thomsen condition. Of the double cancellation studies, three support it: Falmagne, Iverson, & Marcovici (1979), Levelt, Riemersma, & Bunt (1972), and Schneider (1988), where the latter differed from the other studies in having frequencies varying by more than a critical band in the two ears. Rejecting it were Falmagne (1976) with but one respondent, and Gigerenzer and Strube (1983) with 12 respondents. Because of the inconsistent pattern of results, we felt it necessary to test the Thomsen condition within our own experimental context. Our experimental design was closest to that of Gigerenzer and Strube (1988).

The Thomsen condition was tested by successively obtaining the estimates, $z'$, $y'$, and $y''$, in

$$\begin{array}{rcl}(x,t) & \sim & (z',v)\\(z',u) & \sim & (y',t)\\(x,u) & \sim & (y'',v)\end{array}$$

using the trial form in (17), where the first of two tones in the second joint-presentation is varied. The property is said to hold if $y'$ and $y''$ are found to be statistically equivalent.

We used two stimuli sets, **A** and **B**, in our test of the Thomsen condition:

$$\begin{array}{lcl}\mathbf{A} & : & x = 66, t = 62, v = 58, \text{ and } u = 70 \text{ dB},\\\mathbf{B} & : & x = 62, t = 59, v = 47, \text{ and } u = 74 \text{ dB}.\end{array}$$

Stimulus set **B** consisted of stimuli having the same relative intensity relationship as those used by Gigerenzer and Strube (1988), although we used 1,000 Hz whereas they used both 200 Hz and 2,000 Hz, a difference that may be relevant.

We initially ran the respondents on **A**, after which we decided to add **B** in order to have a more direct comparison with their study.

With 12 respondents, there were 24 tests of which 5 rejected the null hypothesis. Of the 5 failures, 4 occurred in set **A** and 1 in **B**. This fact suggests that a good deal of practice may regularize the behavior (see SL-I for details).

## 3.3 Bisymmetry

On the assumption that we have a p-additive representation, (12), we next turn to the question of whether or not $\delta = 0$. All of the experimental testing is a good deal simpler when $\delta = 0$ than it would be otherwise–an example is the testing of the property called joint-presentation decomposition (Sec. 4.1).

Given the p-additive representation, one can then show (Luce, 2004, Corollary 2 to Theorem 1, p. 450) that for people who violate jp-symmetry, (20), then $\delta = 0$ is equivalent to the property:

**Bisymmetry:**

$$(x \oplus_i y) \oplus_i (u \oplus_i v) = (x \oplus_i u) \oplus_i (y \oplus_i v) \quad (i = l, r, s). \tag{24}$$

Note that the two sides of bisymmetry simply involve the interchange of $y$ and $u$. Bisymmetry is not predicted when $\delta \neq 0$ except for constant bias with $\gamma = 1$. Because we have considerable evidence against $\gamma = 1$ (Sec. 3.1), whether bisymmetry holds tells us whether or not $\delta = 0$.

Testing involved obtaining the estimates

$$w_i = x \oplus_i y \text{ and } w_i' = u \oplus_i v, \quad \text{[right side of (24)]},$$
$$z_i = x \oplus_i u \text{ and } z_i' = y \oplus_i v \quad \text{[left side of (24)]}.$$

Then in a second step, obtain

$$t_i = w_i \oplus_i w_i' \text{ and } t_i' = z_i \oplus_i z_i'.$$

The property is said to hold if $t_i$ and $t_i'$ are found to be statistically equivalent. The property was tested using both symmetric and left matches using trials of the form (17) and (15), and intensities $x = 58$ dB, $y = 64$ dB, $u = 70$ dB, and $v = 76$ dB. With 6 respondents there were no rejections of bisymmetry. So we assume $\delta = 0$ in what follows (SL-I).

## 3.4 Production commutativity

If we rewrite (13)[6] as

$$\Psi\left[(x, x) \circ_p (y, y)\right] = W(p)[\Psi(x, x) - \Psi(y, y)] + \Psi(y, y),$$

[6] To those familiar with utility theory, the following form is basically subjective weighted utility (Luce & Marley, 2004).

then by direct substitution the following behavioral property follows.

**Production commutativity:** For $p > 0, q > 0$,

$$[(x,x) \circ_p (y,y)] \circ_q (y,y) \sim [(x,x) \circ_q (y,y)] \circ_p (y,y). \tag{25}$$

Observe that the two sides differ only in the order of applying $p$ and $q$, which is the reason for the term "commutativity." This property also arose in Narens' (1996) theory of magnitude estimation. Ellermeier and Faulhammer (2000) tested that prediction in the special case where $y = 0$ for $p, q > 1$ and Zimmer (2004) did so for $p, q < 1$. Both studies found it sustained. The general form of production commutativity has yet to be tested with $p, q < 1$.

In the presence of our other assumptions, production commutativity turns out to be sufficient for (13) to hold.

Production commutativity was tested using symmetric ratio productions requiring four estimates in two steps. The first consisted of

$$(x,x) \circ_p (y,y) \sim (v,v),$$
$$(v,v) \circ_q (y,y) \sim (w,w),$$

and in the second of

$$(x,x) \circ_q (y,y) \sim (v',v'),$$
$$(v',v') \circ_p (y,y) \sim (w',w').$$

The property is considered to hold if $w$ and $w'$ are found to be statistically equivalent. Trials were of the form in (18). The intensities we used were $x = 64$ dB and $u = 70$ dB and the proportions that we used were $p = 2$ and $q = 3$, giving rise to four trial conditions in each step. Four respondents yielded 4 tests and the null hypothesis of production commutativity was not rejected in any of them (SL-I).

## 3.5 Discussion

The results of the experiments on the Thomsen condition and on proportion commutativity support the forms $\Psi$ in (12) and $\Psi_\Theta$ in (13) separately. Although we have no evidence at this point for assuming that $\Psi_\Theta = \Psi$, we do know that both are strictly increasing with $\succsim$, and so there is a strictly increasing, real-valued function connecting them: $\Psi_\Theta = f(\Psi)$.

So our next task is to ask for conditions necessary and sufficient for the function $f$ to be the identity function. To show this requires some interlocking of the two structures $\langle \mathbb{R}_+ \times \mathbb{R}_+, \succsim \rangle$ and $\langle \mathbb{R}_+ \times \mathbb{R}_+, \succsim, \circ_p \rangle$, which were reduced to the one dimensional structures of the form $\langle \mathbb{R}_+, \geq, \oplus_i \rangle$ and $\langle \mathbb{R}_+, \geq, \circ_{p,i} \rangle$, respectively. We now turn to that interlocking issue.

# 4 Links Between Summation and Production

It turns out that two necessary properties of the representations establish the needed interlock or linkage between the primitives, and these properties along with those discussed earlier are sufficient to yield a common representation (Theorem 2 of Luce, 2004). In a sense, the novelty of the present theory lies in formulating their interlock purely behaviorally.

The links that we impose are analogues to familiar “distribution” properties such as those in set theory, namely,

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C), \tag{26}$$

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C). \tag{27}$$

If we replace $\cup$ by $\oplus$ and $\cap$ by $\circ_p$ we get, respectively, what are called below simple joint-presentation decomposition (Sec. 4.1) and segregation (4.2).

To help formulate these properties, the following induced production operations $\circ_{p,i}$, $i = l, r, s$, which are special cases of the general operation $\circ_p$ defined by (10), are introduced:

$$(x \circ_{p,l} y, 0) \quad : \quad = (x, 0) \circ_p (y, 0), \tag{28}$$

$$(0, u \circ_{p,r} v) \quad : \quad = (0, u) \circ_p (0, v), \tag{29}$$

$$(x \circ_{p,s} y, x \circ_{p,s} y) \quad : \quad = (x, x) \circ_p (y, y). \tag{30}$$

## 4.1 Simple joint-presentation decomposition

As suggested above, the analogue of (26), linking the two operations $\oplus_i$ and $\circ_{p,i}$ is:

**Simple Joint-Presentation (SJP-) Decomposition:**

$$(x \oplus_i u) \circ_{p,i} 0 = (x \circ_{p,i} 0) \oplus_i (u \circ_{p,i} 0) \quad (i = l, r, s). \tag{31}$$

*Comment:* When $\delta \neq 0$, the corresponding property becomes vastly more complex to test because the term $u \circ_{p,i} 0$ is replaced by $u \circ_{q,i} 0$ where $q = q(x, p)$. Thus, one must first determine $q(x, p)$ empirically and then the condition corresponding to (31) with $q$ replacing the second $p$ on the right is checked.

Simple JP-decomposition has two levels of estimation which were done in two steps. First, the estimates

$$\begin{aligned} t_s &= (x \oplus_s u) \circ_{p,s} 0, \\ w_s &= x \circ_{p,s} 0, \\ s_s &= u \circ_{p,s} 0, \end{aligned}$$

were obtained using trials of the form in (18). The averages of $w_s$ and $s_s$ were used in the second step, which consisted of the match

$$t'_s = w_s \oplus_s s_s,$$

using trials of the form in (16). The property is considered to hold if $t_s$ and $t'_s$ are found to be statistically equivalent. We used one pair of intensities, $x = 64$ dB and $u = 70$ dB. With 4 respondents there were 8 tests and SJP-decomposition was not rejected in 6 (SL-II).

## 4.2 Segregation

The second property linking the two operations, the analogue of (27) but taking into account the non-commutativity of $\oplus_i$, is:

**Segregation:** For all $x, u, p \in \mathbb{R}_+$,

**Left segregation:**

$$u \oplus_i (x \circ_{p,i} 0) \sim (u \oplus_i x) \circ_{p,i} (u \oplus_i 0) \quad (i = l, r, s). \tag{32}$$

**Right segregation:**

$$(x \circ_{p,i} 0) \oplus_i u \sim (x \oplus_i u) \circ_{p,i} (0 \oplus_i u) \quad (i = l, r, s). \tag{33}$$

If jp-symmetry, (20), holds, then right and left segregation are equivalent. Otherwise they are distinct.

Note that because $0$ is a right identity of $\oplus_l$, (7), testing left segregation is easier for $i = l$, and, similarly, right segregation is easier for $i = r$. For $i = s$ both need to be tested.

For each respondent (except one), we studied only one form of segregation, either left or right (see SL-II for details).

In the case of right segregation, four estimates must be made

$$\begin{aligned} w_r &= x \circ_{p,r} 0, \\ t_r &= w_r \oplus_r u, \\ z_r &= x \oplus_r u, \\ t'_r &= z_r \circ_{p,r} u. \end{aligned}$$

The property is said to hold if $t_r$ and $t'_r$ are not found to be statistically different.

Note that the intensities $w_r$ and $z_r$ are first estimated in the right ear but then they must be presented in the left ear for the case $(w_r, u) \sim (0, w_r \oplus_r u)$. The converse is true for left segregation. The trials used for matching were of the forms in (15–17) depending on the matching ear. Symmetric ratio productions were obtained using (18) and (19) or their equivalent asymmetric productions.

We used one intensity pair, $x = 72$ dB and $u = 68$ dB, except for one respondent where each was decreased by $4$ dB to avoid productions limited by a $85$ dB safety bound. A theoretical predication is that the property holds for both $p < 1$ and $p \geq 1$, hence $p = 2/3$ and $p = 2$ were used.

Four respondents produced 10 tests and the null hypothesis was accepted in 8 of them (SL-II).

### 4.3 Discussion

Given the complexity of testing these two properties of the model and given the potential for artifacts, we feel that the support found for the model leading to the additive representation, (12) with $\delta = 0$, and the subjective proportion representation, (13), is not too bad.

Assuming that there is a common $\Psi$ underlying the representations, an interesting theoretical challenge exists. Taken by itself, the p-additive representation of $\oplus_i$ could have $\delta \lesseqgtr 0$. For $\delta < 0$, it is not difficult to see that $\Psi$ is bounded by $1/|\delta|$ and so $\Psi : \mathbb{R}_+ \times \mathbb{R}_+ \overset{onto}{\longrightarrow} I = [0, 1/|\delta|]$. (Of course, we have data on bisymmetry that suggests $\delta = 0$. However, we cannot really rule out that $\delta$ may be slightly different from $0$.) On the face of it, boundedness seems quite plausible. Psychophysical scales of intensity seem to have upper bounds tied in with potential sensory damage and so infinite ones are decidedly an idealization. However, bounded $\Psi$ is definitely not possible[7] because one can repeatedly iterate the operator as, e.g., in the second step: $(x \circ_{p,i} y) \circ_{p,i} y$. This forces $\Psi$ to be unbounded. So the challenge is to discover a suitable modification of (13) that is bounded and work out its properties.

## 5 Sensory Filtering, Multiplicative Invariance, and Forms for $\Psi$

### 5.1 Asymmetric matching and jp-symmetry

At the beginning of Sec. 3 we used symmetric matches to check joint-presentation symmetry and left until now the use of asymmetric left and right matches. They sometimes exhibited the following phenomenon which at first seemed disturbing but, in fact, seems to have rather mild consequences. Consider the asymmetric matches

$$\begin{aligned}(x,u) &\sim (x \oplus_l u, 0),\ (u,x) \sim (u \oplus_l x, 0),\\ (x,u) &\sim (0, x \oplus_r u),\ (u,x) \sim (0, u \oplus_r x).\end{aligned}$$

Suppose that jp-symmetry fails, as it often seems to, and suppose that $(x,u) \succ (u,x)$. Then one expects to observe that both $x \oplus_l u > u \oplus_l x$ and $x \oplus_r u > u \oplus_r x$–that left and right matches will agree in what they say about jp-symmetry. We carried out such an experiment, obtaining the asymmetric matches above using the trial forms in (15) and (16) and the same stimuli as in Sec. 3.1. Although the expected agreement held for 4 respondents, it did not for 2, even after considerable experience in the experimental situation. Moreover, for those who were qualitatively consistent in the sense above, the magnitude of the differences $x \oplus_l u - u \oplus_l x$ and $x \oplus_r u - u \oplus_r x$ varied considerably. Evidently, matching in a single ear had some significant impact. Of course, one impact manifest itself in a sharp change of localization, which at first seems a bit odd to people. But the inconsistency just described is more worrisome for our experiments. This means that an experimental procedure that relies on the assumption of bias independent

[7]This fact was pointed out by Ehitbar Dzhafarov in a personal communication in his role as referee of Luce (2004).

of the matching ear will not be reliable. In practice this has not proven to be an obstacle in our other experiments. We offer a possible account of this effect in the next two subsections.

## 5.2 Sensory filtering in the asymmetric cases

Suppose that asymmetric matching has the effect of either enhancing (in the brain) all signals in the matching ear or attenuating those in the other ear. If these effects entail a simple multiplicative factor on intensity, i.e., a constant dB shift, then the two ideas are equivalent. If we assume that there is an attenuation or filter factor $\eta$ on the non-matched ear, then for the left matching case the experimental stimulus $(x, u)$ becomes, effectively, $(x, \eta u)$. And when matching in the right ear, $(x, u)$ it becomes effectively $(\eta x, u)$, where $0 < \eta \leq 1$. Thus, when we ask the respondents to solve the three indifferences of (3) what they actually do, according to this theory, is set

$$\begin{aligned} z_l &= x \oplus_l \eta u \Leftrightarrow (z_l, 0) \sim (x, \eta u), \\ z_r &= \eta x \oplus_r u \Leftrightarrow (0, z_r) \sim (\eta x, u), \\ z_s &= x \oplus_s u \Leftrightarrow (z_s, z_s) \sim (x, u). \end{aligned}$$

Note that the filter plays no role in the symmetric matches.

Under a further condition called multiplicative invariance (Sec. 5.3), which is equivalent to $\delta = 0$ and that $\Psi(x, 0)$ and $\Psi(0, x)$ are each a power function of $x$, but with different powers, one can show that the filtering concept does indeed accommodate the above phenomenon of asymmetric matching in connection with checking jp-symmetry.

## 5.3 Multiplicative invariance

Fortunately, we were able to show that filtering did not distort any of the experimental tests of the properties of Secs. 3 and 4 where asymmetric matching is used, provided that the operations $\oplus_i$ have an additive representation (shown in Secs. 3.2 and 3.3) and that the following property holds:

$\sigma-$**Multiplicative Invariance ($\sigma-$MI):** For all signals $x \geq 0, u \geq 0$, for any factor $\lambda \geq 0$, and for $\oplus_i$, $i = l, r$, defined by (3) and (4), there is some constant $\sigma > 0$ such that:

$$\lambda x \oplus_l \lambda^\sigma u = \lambda(x \oplus_l u), \tag{34}$$

and

$$\lambda x \oplus_r \lambda^\sigma u = \lambda^\sigma (x \oplus_r u). \tag{35}$$

We observe that this property is, itself, invariant under sensory filtering because with filtering that expression becomes

$$\begin{aligned} \lambda x \oplus_l \eta \lambda^\sigma u &= \lambda(x \oplus_l \eta u), \\ \eta \lambda x \oplus_r \lambda^\sigma u &= \lambda^\sigma (\eta x \oplus_r u). \end{aligned}$$

Using the fact that $\eta \lambda^\sigma = \lambda^\sigma \eta$ and setting $v = \eta u$ or $y = \eta x$ yields (34) and (35) with a trivial change of notation. So the filter does not affect any further uses of $\sigma-$MI.

Turning to our other necessary properties in Secs. 3 and 4, elementary calculations show that they are invariant under filtering either with no further assumption or assuming multiplicative invariance, see Table 1:

| | Assumption | |
|---|---|---|
| Property | None | $\sigma-$MI |
| Thomsen | X | |
| Bisymmetry | | X |
| Prod. Comm. | X | |
| SJP-Decomp. | | X |
| Segregation | | X |

Table 1: Effect of filtering on properties.

We examine one important implication of $\sigma-$MI in the next subsection and report some relevant data.

### 5.4 $\Psi$ a sum of power functions

So far, we have arrived at a representation with two free parameters, $\delta$ and $\gamma$, and two free increasing functions, $\Psi$ and $W$, and shown that, most likely, $\delta = 0$. It is clear that one further goal of our project is to characterize behaviorally each of the functions to a specific family with very few free parameters. In this section we take up one argument for $\Psi$ being a sum of power functions and in Sec. 6.1 we give a different argument for the power function form of $\Psi$ and also consider two possible forms for $W$, rejecting one and possibly keeping the other.

Assuming that the representation (12) holds (see Secs. 1.3 and 3.2) and that $\delta = 0$ (see Sec. 3.3), then one can show that multiplicative invariance is equivalent to $\Psi$ being a sum of power functions, (48), with exponents $\beta_l$ and $\beta_r$ such that $\sigma = \beta_l/\beta_r$, i.e.

$$\Psi(x,u) = \alpha_l x^{\beta_l} + \alpha_r u^{\beta_r} = \alpha_l x^{\beta_l} + \alpha_r u^{\beta_l/\sigma}. \tag{36}$$

The proof is a minor modification of that given by Aczél, Falmagne, and Luce (2000, Sec. 2.2.1) for $\sigma = 1$. Thus, multiplicative invariance is a behavioral test for the power function form (36).

Note that

$$\frac{\Psi(x,0)}{\Psi(0,x)} = \frac{\alpha_l}{\alpha_r} x^{\beta_l - \beta_r}.$$

Thus the constant bias property (14) holds iff $\gamma = \frac{\alpha_l}{\alpha_r}$ and $\sigma = \frac{\beta_l}{\beta_r} = 1$.

Recall that $x$ and $u$ in (34) and (35) are intensity differences between the signal intensity actually presented and the threshold intensity for that ear. However, the experimental design and results are typically reported in dB terms. In the current situation, this practice represents a notational difficulty because, e.g. $\lambda x$ in dB terms is:

$$10\log(\lambda x) = 10\log\lambda + 10\log x.$$

Thus, in dB the multiplicative factor is additive. In the following, the intensity notation will be maintained in equations but actual experimental quantities will be reported in dB's where $\lambda_{dB} = 10\log\lambda$ stands for the additive factor.

In order to test multiplicative invariance, it is most desirable to estimate $\sigma$ and not to have to run a parametric experiment. To that end, using the representation (36) one can show that

$$(0 \oplus_l x)_{dB} = c_1 + \frac{1}{\sigma} x_{dB}, \tag{37}$$

$$(x \oplus_r 0)_{dB} = c_2 + \sigma x_{dB}, \tag{38}$$

from which

$$(x \oplus_r 0)_{dB} = c_3 + \sigma^2 (0 \oplus_l x)_{dB} \tag{39}$$

follows. This appears to be a suitable way to estimate $\sigma$–suitable in the sense that if (36) holds, then this is what it must be.

In terms of the power function representation itself one can show that the constants $c_1$ in (37), $c_2$ in (38), and $c_3$ in (39) are explicit functions of $\gamma$ and $\eta$ and, solving for these parameters, one can show that

$$\log\eta = \frac{\sigma c_1 + c_2}{10(1+\sigma)}, \tag{40}$$

$$\log\gamma = \frac{\beta_r c_3}{10(1+\sigma)}. \tag{41}$$

## 5.5 Tests of $1-$MI

We did this experiment before we had developed the general result about $\sigma-$MI. The test was carried out in two steps: The first is an experimental one in which the respondents estimate

$$t_i = (\lambda x) \oplus_i (\lambda u) \text{ and } z_i = x \oplus_i u,$$

obtained using trial-form (15) or (16) as the case might be. This is followed by a purely "arithmetic" step in which the multiplication $t'_i = \lambda \times z_i$ is performed by the experimenter. Multiplicative invariance is said to hold if $t_i$ and $t'_i$ are found statistically equivalent.

For the experiment we used $x = 64$ dB and $u = 70$ dB and two values for $\lambda_{dB}$, $4$ and $-4$ dB ($\lambda = 2.5$ and $0.4$, respectively).

Of 22 respondents, 12 satisfied multiplicative invariance in both tests, three failed both, and seven failed one. So we have a crude estimate of about half of the respondents satisfying multiplicative with $\sigma = 1$. The fact of so many failures led us to explore how to estimate $\sigma$ and then to check multiplicative invariance using that estimate.

## 5.6 Estimating $\sigma$

For 7 of the respondents for whom we tested $1-$MI, we also collected the estimates $z_r = (x \oplus_r 0)$ and $z_l = (0 \oplus_l x)$ using trial-forms (15) and (16), and the three instantiations of $x$, 58, 66, and 74 dB SPL. Then, using linear regression, we obtained an estimate for $\sigma$ and statistically tested whether $\sigma = 1$. These results, including the numerical direction of the estimated $\sigma$'s, for the testing of 7 respondents are summarized Table 2.

| Result for $\sigma-$MI | Total | Statistics | | Numerical | |
|---|---|---|---|---|---|
| | | $\sigma = 1$ | $\sigma \neq 1$ | $\sigma < 1$ | $\sigma > 1$ |
| Passed | 5 | 3 | 2 | 4 | 1 |
| Failed | 2 | 2 | 0 | 1 | 1 |

Table 2: Summary of $\sigma$-estimation results.

There seems to be some limited evidence of gaining a little, but not a lot, by going to $\sigma-$MI.

To evaluate these results further we examined the previous data testing $1-$MI (22 respondents) and asked in what direction would $\sigma$ have to deviate from $1$ in order not to reject $\sigma-$MI. These results are summarized Table 3.

| Result for $\sigma-$MI | Total | $\sigma < 1$ | $\sigma > 1$ | contradictory |
|---|---|---|---|---|
| Passed | 12 | 2 | 7 | 3 |
| Failed | 10 | 2 | 7 | 1 |

Table 3: Summary of direction needed to fit data.

The contradictory results occur when one of the $1-$MI conditions passed robustly, and so it may just be an issue of variance.

What is immediately striking is that for 14/22 respondents we anticipate a value of $\sigma > 1$, whereas for the subset of 7 respondents for which we estimated of $\sigma$ (Table 2) 5/7 had estimates of $\sigma < 1$. For these 7 respondents, $\sigma > 1$ was expected for 5 respondents, $\sigma < 1$ for 1, and was contradictory for 1 respondents. In only one case was the expected numerical direction of $\sigma$ the same as the one obtained. This means the pattern of results appears wrong in 5/7 cases.

In conclusion, the results of the $\sigma$ estimation do not seem to provide a correction factor that explains the respondents deviation from $1-$MI. Thus, we have evidence for about 50% of respondents being well described by the sum of power functions, but we do not know what forms fit the other half.

## 6 Ratio Estimation and the Forms for $W$

To those familiar with the empirical literature on "direct scaling" methods, our discussion may seem unusual because so far it has focused exclusively on ratio production and not at all on ratio estimation and its close relative magnitude estimation. Magnitude estimation is far more emphasized in the empirical and applications literatures than is magnitude production. We remedy this lacuna in the theory now.

Here it is useful to define:

$$\psi_l(x) := \Psi(x, 0), \tag{42}$$
$$\psi_r(u) := \Psi(0, u), \tag{43}$$
$$\psi_s(x) := \Psi(x, x). \tag{44}$$

We will work with the generic $\psi_i$, $i = l, r, s$.

### 6.1 Ratio estimation interpreted within this theory

A fairly natural interpretation of ratio estimations can be given in terms of (13) with $y = 0$. Instead of producing $z_i(x, p) = x \circ_{p,i} 0$, $i = l, r, s$, such that $z_i(x, p)$ stands in the ratio $p$ to $x$, the respondent is asked to state the value of $p_i$ that corresponds to the subjective ratio of $z$ to $x$. This value may be called the *perceived ratio* of intensity $z$ to intensity $x$. If we change variables by letting $t = z/x$, then $p_i$ is a function of both $t$ and $x$, i.e., $p_i = p_i(t, x)$. Note that $p_i$ is a dimensionless number. According to (13) and using the definition of $\psi_i$,

$$W\left(p_i(t, x)\right) = \frac{\psi_i(tx)}{\psi_i(x)}. \tag{45}$$

This relation among the three unknown functions, $\psi_i, p_i, W$, is fundamental to what follows.

The empirical literature on magnitude estimates has sometimes involved giving a standard $x$ and in some experiments it was left up to the respondent to set his or her own standard. Stevens (1975, p. 26-30) argued for the latter procedure. From our perspective, this means that it very unclear what a person is trying to do when responding–comparing the present stimulus with some fixed internal standard or to the previous signal or to what? And, therefore, it means that averaging over respondents, who may be doing different things, is even less satisfactory than it usually is.

The literature seems to have assumed implicitly that the ratio estimate $p_i(t, x)$ depends only on $t$, not on $x$, i.e.,

$$p_i(t, x) = p_i(t). \tag{46}$$

The only auditory data we have uncovered on this is in Hellman and Zwislocki (1961). They had 9 respondents provide ratio estimates to five different standard pairs $(x_0, 10)$ where $x_0 = 40, 60, 70, 80, 90$ dB SPL. The geometric-mean results for the respondents are shown in their Fig. 6. If one shifts the intensity scale (in dB) so that all the standard pairs are at the same point of the graph, we get the plot shown in the left panel of Fig. 1. For values above the standard, there does not seem to be any difference in the curves, in agreement with (46). But things are not so favorable for values below the standard. Of course, there are possible artifacts. Experience in this area suggests that many people are uneasy about the lower end of the numerical scale, especially below $1$. They seem to feel "crowded" in the region of fractions. Also, in our theory one should treat the abscissa as the intensity less the threshold intensity, which they had no reason to do.

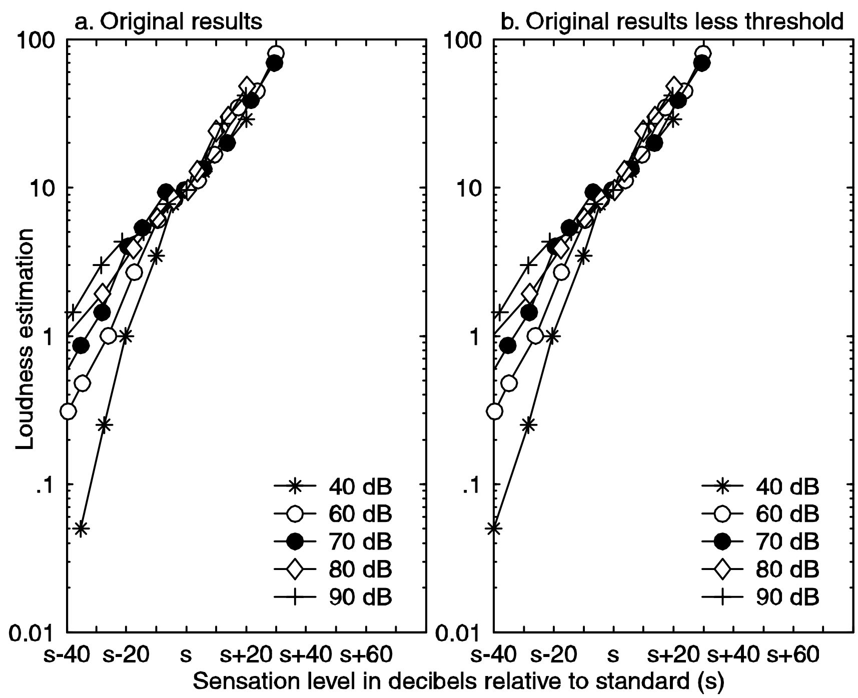


Figure 1: Adapted from the data in Fig. 6. of Hellman and Zwitlocki (1961)

They reported an average threshold of 6 dB SPL. The right panel of Fig. 1 shows such a correction. It does not materially affect the failure of (46) for $p < 1$.

One would like to see their study redone with two changes: First, set the standards at $(x_0, 100)$. Second, collect data on individual thresholds and plot the data in terms of the intensity less the threshold intensity for individual respondents.

## 6.2 Psychophysical power functions

Anyhow, assuming that (45) holds, then (46) immediately yields

$$W(p_i(t)) = \frac{\psi_i(tx)}{\psi_i(x)}, \tag{47}$$

which is a Pexider functional equation (Aczél, 1966, p. 144) whose solutions with $\psi_i(0) = 0$ are, for some constants $\alpha_i > 0, \beta_i > 0$,

$$\begin{aligned} \psi_i(t) &= \alpha_i t^{\beta_i} \quad (t \geq 0), & (48) \\ W[p_i(t)] &= t^{\beta_i} \quad (t \geq 0). & (49) \end{aligned}$$

Recall that the $\psi_i$ are the production psychophysical functions defined in terms of $\Psi$ by (42) for $i = l$ and by (43) for $i = r$. So (48) agrees with our earlier result about sums of power functions being implied when multiplicative invariance is satisfied, (Sec. 5.4). And, of course, (46) holds if the psychophysical function is a power function.

## 6.3 Do ratio estimates also form power functions?

The conclusion (49) tells us that, when we observe empirically the estimation function $p_i(t)$, it is a power function, but it is seen through the distortion $W^{-1}$. Stevens (1975) claimed that the magnitude estimation psychophysical functions are, themselves, power functions, which was approximately true for geometric means over respondents; however this is not really the case for data collected on individuals. See Fig. 2. (This is Fig. 1, p. 292, of Green & Luce, 1974). This fact is again a caution about averaging over respondents.

Moreover, Stevens (1975) attempted to defend the position that both the magnitude and production functions are power functions although he was quite aware that empirically they do not prove to be simple inverses of one another (p. 31). Indeed, he spoke of an unexplained "regression" effect which has never really been fully illuminated (Stevens, 1975, p. 32).

So let us consider the possibility that, as Stevens claimed,

$$p_i(t) = \rho_i t^{\nu_i} \quad (t > 0, \rho_i > 0, \nu_i > 0). \tag{50}$$

Note that because $p_i$ is dimensionless, the parameter $\rho_i$ is a constant, not a free parameter. It is quite easy to see that if (49) holds, then $p_i$ is a power function, (50), if, and only if, $W(p)$ is also a power function with exponent $\omega_i := \beta_i/\nu_i$, i.e.,

$$\begin{aligned} W_i(p) &= \left(\frac{p}{\rho_i}\right)^{\omega_i} \\ &= W_i(1)p^{\omega_i} \quad (p \geq 0). & (51) \end{aligned}$$

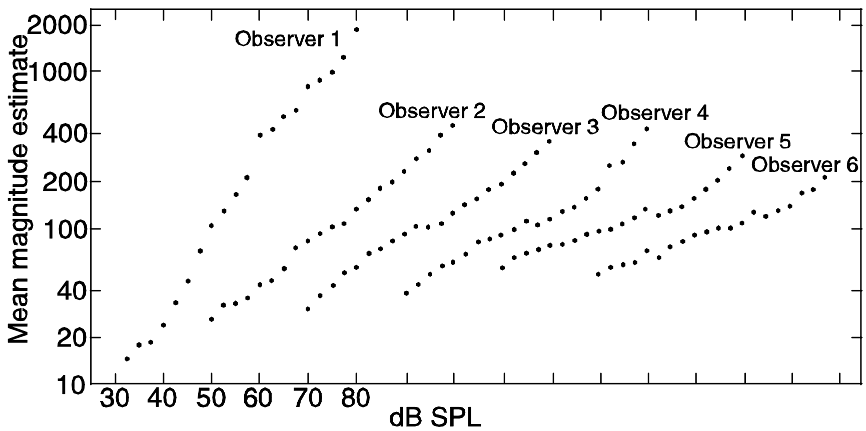


Figure 2: Reproduction of Fig 1 of Luce and Green (1974)

This form has different implications depending on whether the constant $\rho_i = 1$ or $\neq 1$. Note that $\rho_i = 1$ holds iff $W_i(1) = 1$. From here on we assume that $W_i$, being a cognitive function, is independent of $i = l, r$ and so can be denoted $W$. Both cases rest on an exploration of the property of *threshold production commutativity:*

$$(x \circ_{p,i} 0) \circ_{q,i} 0 = (x \circ_{q,i} 0) \circ_{p,i} 0 = x \circ_{t,i} 0, \tag{52}$$

which by (13), is equivalent to

$$W(p)W(q) = W(t). \tag{53}$$

To increase generality, we suppose that (53) holds for $p > 1, q > 1$ and, separately, for $p < 1, q < 1$, but not necessarily for the crossed cases: $p > 1 > q$ or $q > 1 > p$. Assuming the continuity of $W(p)$ at $p = 1$, it is easy to show that if this obtains the following statements are equivalent: (1) There exist constants $\omega_>$ and $\omega_<$ such that

$$W(p) = W(1) \begin{cases} p^{\omega_>}, & p \geq 1 \\ p^{\omega_<}, & p < 1 \end{cases} . \tag{54}$$

(2) The relation among $p, q$, and $t$ is given by:

$$t = pq \begin{cases} W(1)^{1/\omega_>}, & p \geq 1 \\ W(1)^{1/\omega_<}, & p < 1 \end{cases} . \tag{55}$$

If we also assume that (53) holds for $p > 1 > q$ or $p < 1 < q$, then $\omega_> = \omega_<$. Some pilot data we have collected strongly suggests that (55) does not hold for the crossed cases $p > 1 > q$ and $p < 1 < q$ and that $W(1) < 1$. Further empirical work will be reported in SL-IV.

The only published data we are aware of concerning (52) is that of Ellermeier and Faulhammer (2000) and Zimmer (2004). They restricted their attention to the case of $\rho_i = 1$ which is equivalent to $W(1) = 1$. Narens (1996) arrived at $pq = t$ as a consequence of his formalization of what he believed Stevens (1975) might have meant theoretically when invoking magnitude methods. Ellermeier and Faulhammer (2000) and Zimmer (2004) tested (55) experimentally and unambiguously rejected it. To our knowledge no one has attempted to collect sufficient data to see how well (55) fits the data with $W(1) \neq 1$. We do know from Ellermeier and Faulhammer (2000) that, on average, $t < pq$, which means that $W(1) < 1$, on average, which was also supported by our pilot data sited above. This has the following implication. If $(x, x)$ and $(y, y)$ are presented and a respondent is asked to produce $z$ such that the interval from $(y, y)$ to $(z, z)$ is the same as the interval from $(y, y)$ to $(x, x)$, they will select $z < x$. This has yet to be checked empirically.

So the answer to the question of the heading—Do ratio estimates form power functions?—is that at this point we do not know. The key prediction (55) has not yet been checked. If, however, the general power function form is rejected, then the task of finding the form of $W$ remains open. We discuss next one interesting, but ultimately unsuccessful, attempt, the Prelec function.

## 6.4 If ratio estimation is not a power function, what is $W$?

### 6.4.1 Prelec's function

Within the context of utility theory for risky gambles and for $0 < p \leq 1$, a weighting function was proposed and axiomatized by Prelec (1998) that had a desirable feature. Depending on the combinations of the parameters, the function is concave, convex, S-shaped, or inverse S-shaped. Empirical data on preferences among gambles seemed to suggest that the inverse S-shaped form holds (Luce, 2000, Sec. 3.4, especially Fig. 3.10 on p. 99). The Prelec form for the weighting function, generalized from the unit interval to all positive numbers is

$$W(p) = \begin{cases} \exp\left[-\lambda\left(-\ln p\right)^{\mu}\right] & (0 < p \leq 1) \\ \exp\left[-\lambda'\left(\ln p\right)^{\mu}\right] & (1 < p) \end{cases}, \qquad (56)$$

where $\lambda > 0$, $\lambda' > 0$, and $\mu > 0$. The special case of $\mu = 1$ is a power function with $W(1) = 1$, which we know is wrong.

### 6.4.2 Reduction invariance: a behavioral equivalent of Prelec's $W$

Prelec gave one axiomatization of the form (56) and Luce (2001) gave the following simpler one:

**Reduction Invariance:** Suppose that positive $p, q, t = t(p, q)$ are such that (52) is satisfied for all $x > 0$. Then for any natural number $N$,

$$(x \circ_{p^N,i} 0) \circ_{q^N,i} 0 = x \circ_{t^N,i} 0. \qquad (57)$$

In words, if the compounding of $p$ and $q$ in magnitude productions is the same as the single production of $t$, then the compounding of $p^N$ and $q^N$ is the same as the single production of $t^N$. On the assumptions that (53) holds for $p^N$, $q^N$, and $t^N$ and that $W$ is strictly increasing function on the interval $]0,1]$, Luce (2001) showed that reduction invariance, (57), is equivalent to the Prelec function (56) holding in the unit interval. Indeed, it turns out that its holding for two values of $N$ such as $N = 2, 3$ are sufficient to get the result. Another pair that works equally well is $N = 2/3, 2$. It is not difficult to see how to extend the proof to deal with the interval $]1, \infty[$. One can also show that it works for $N$ any positive real number; however, any two values without a common factor suffice.

Zimmer (2004) was the first to test this hypothesis and she rejected it. Her method entailed working with bounds and showing that the observed data falls outside them. In SL-IV, we also tested it using our ratio-production procedure. We too found that it failed. The fact that $W$ is a cognitive distortion of numbers may mean that it will also fail empirically in other domains, such as utility theory, when reduction invariance is studied directly.

Testing was done using two-ear ($i = s$) productions. First, the two successive estimates

$$v_s = x \circ_{p,s} 0, \tag{58}$$

$$t_s = z_s \circ_{p,s} 0, \tag{59}$$

were obtained. Then, using the simple Up-Down method (Levitt, 1971), $t$ was estimated such that $x \circ_{t,i} 0 \sim t_s$. With the estimate of $t$ and $N$ the following estimates were obtained:

$$t'_s = (x \circ_{p^N,i} 0) \circ_{q^N,i} 0$$
$$w'_s = x \circ_{s^N,i} 0.$$

The property is said to hold if $t'_s$ and $w'_s$ are found statistically equivalent.

We used the two instantiations, $x = 64$ dB and $x = 70$ dB, and the proportions, presented as percentages, $p = 160\%$ and $q = 80$, except for one respondent where $q = 40\%$ another for another where $p = 140\%$. The power $N$ was chosen as close to 2 as would provide numbers close to a multiple of five for each of $p^N$, $q^N$, and $t^N$.

The property was rejected for six of six respondents. For three the failure was beyond much question. But taking into account the complexity of the testing procedure and the multiple levels of estimation, the failure for the other three was not dramatic. Indeed, had our data been as variable as Zimmer's (2004), we almost certainly would have accepted the property of reduction invariance in those three cases.

When we tested reduction invariance, we did not know about the potential problems of testing this property using the mixed case of $p > 1$, $q < 1$ outlined in Section 6.3. Without further testing, the failure we observed is potentially related to this issue, however Zimmer's (2004) data are not based on mixed cases; she used $p < 1$, $q < 1$. This further suggests the property should be tested with $p > 1$, $q > 1$; we aim to report such data in SL-IV.

### 6.5 Predictions about co-variation and sequential effects

Given that $\psi_i$ is a power function, we have the following inverse relationships about ratio productions and estimates:

$$r_i(p) = W(p)^{1/\beta_i} \quad (p \text{ given}), \tag{60}$$
$$p_i(r) = W^{-1}(r^{\beta_i}) \quad (r \text{ given}). \tag{61}$$

In the usual dB form in which data are plotted these are

$$r_{i,dB}(p) = \frac{1}{\beta_i} W_{dB}(p), \tag{62}$$
$$p_{i,dB}(r) = W_{dB}^{-1}(r^{\beta_i}) = W_{dB}^{-1}\left(\exp\frac{1}{10}\left(\beta_i r_{dB}\right)\right). \tag{63}$$

#### 6.5.1 What happens when $W$ is a power function?

First suppose that $W$ is a power function of the form (51). Then a routine calculation yields

$$W^{-1}(r^{\beta_i}) = \rho_i r^{\nu_i},$$

and so

$$r_{i,dB}(p) = \frac{1}{\upsilon_i}\left(p_{dB} - \rho_{i,dB}\right),$$
$$p_{i,dB}(r) = \nu_i r_{dB} + \rho_{i,dB}.$$

In response to overwhelmingly clear empirical evidence, several authors have formulated sequential models in which the response in dB on trial $n$, $10\log R_n$, depends linearly on the present signal in dB, $10\log S_n$, the previous one, $10\log S_{n-1}$, the previous response $10\log R_{n-1}$, and in some cases $10\log S_{n-2}$ (DeCarlo, 2003; DeCarlo & Cross, 1990; Jesteadt, Luce, and Green, 1977; Lacouture, 1997; Lockhead, 2004; Luce & Steingrimsson, 2003; Mori, 1998; Petrov & Anderson, 2005)[8]. Both Lockhead and Petrov and Anderson provide many other references to the literature. Setting

$$r_{s,n} = \frac{S_n}{S_{n-1}}, p_{s,n} = \frac{R_n}{R_{n-1}},$$

then each weighting function yields a sequential model for estimation. With symmetric stimuli $(x, x)$, we see that for power functions

$$R_{n,dB} = R_{n-1,dB} + \nu_s\left(S_{n,dB} - S_{n-1,dB}\right) + \rho_{i,dB}.$$

[8]We thank A. A. J. Marley for supplying some of these references.

### 6.5.2 What happens when $W$ is a Prelec function

If we assume that $W$ is given by (56), then putting that form into the expressions for (62) and (63), doing a bit of algebra, and defining $\tau_i := \frac{1}{\beta_i}\left(\frac{\log 10}{10}\right)^{\mu-1}$ yields the following forms for $r_i(p)_{dB}$ and $p_i(r)_{dB}$, respectively:

$$r_i(p)_{dB} = \tau_i \begin{cases} -\lambda\,(-p_{dB})^{\mu} & 0 < p \leq 1 \\ \lambda'\,(p_{dB})^{\mu_i} & 1 < p \end{cases}, \tag{64}$$

$$p_i(r)_{dB} = \frac{1}{\tau_i^{1/\mu}} \begin{cases} -\left(-\frac{\beta_i}{\lambda} r_{dB}\right)^{1/\mu} & 0 < r \leq 1 \\ \left(\frac{\beta_i}{\lambda'} r_{dB}\right)^{1/\mu} & 1 < r \end{cases}. \tag{65}$$

When $\mu$ is approximately 1, then $r_i(p)$ and $p_i(r)$ are approximately power functions, i.e., $r_i(p)_{dB}$ and $p_i(r)_{dB}$ are approximately linear, but with a change of the power at $p = 1$ and $r = 1$, respectively. Of course, it cannot be exactly a power function without contradicting the data of Ellermeier and Faulhammer (2000) and Zimmer (2004). Some examples of (64) are shown in Fig. 3. Such functions seem consistent with the data reported in Fig. 1.

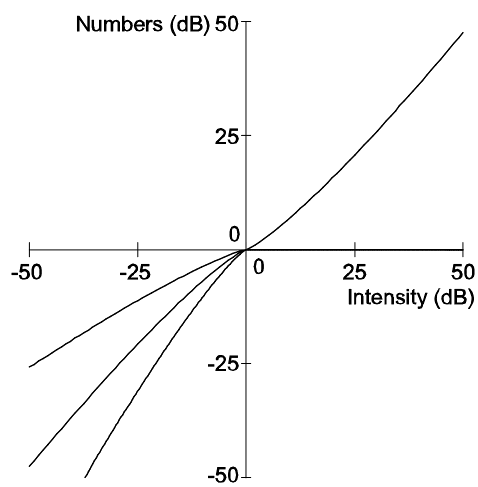


Figure 3. $\psi(x) = \begin{cases} (.5x)^{1.2} & \text{if} \quad 0 \leq x < 50 \\ -(-.cx)^{1.2} & \text{if} \quad -50 < x < 0, \end{cases}$
were $c = 0.3, 0.5, 0.7$ from top to bottom

Proceeding as with the power function but using Prelec's function we get the following predicted sequential effects:

$$R_{n,dB} = R_{n-1,dB} + \tau_s^{1/\mu} \begin{cases} -\left[-\frac{\beta_s}{\lambda}\left(S_{n,dB} - S_{n-1,dB}\right)\right]^{1/\mu} & S_{n,dB} \leq S_{n-1,dB} \\ \left[\frac{\beta_s}{\lambda'}\left(S_{n,dB} - S_{n-1,dB}\right)\right]^{1/\mu} & S_{n,dB} > S_{n-1,dB} \end{cases}.$$

Some aspects of Stevens magnitude estimation and production functions can be illuminated by these results. Let us assume that when the experimenter provides no reference signal $x$, each respondent selects his or her own. Thus, the usual data, which are averaged over the respondents, is the average of approximately piece-wise linear functions with the break occurring in different places. Although (60) and (61) are perfect inverses, it is no surprise that under such averaging the results are not strict inverses of one another. Something like this may provide an account of Stevens' "regression" phenomenon.

# 7 Summary and Conclusions

## 7.1 Summary of the theory

The theory has three primitives:

1. The (loudness) ordering $\succsim$ on $\mathbb{R}_+ \times \mathbb{R}_+$, where $\mathbb{R}_+$ is the set of non-negative numbers corresponding to signals which are intensities less threshold intensity (intensities less than the threshold are subsumed as $0$).

2. The presentation of signal pairs, $(x, u)$, to (the two ears of) the respondent with the defined matching operations $\oplus_i$; and

3. Judgments of "interval" proportions, $\circ_p$.

Within the fairly weak structural assumptions of the theory, necessary and sufficient properties were stated that yield the representations: There exist a constant $\delta \geq 0$ and two strictly increasing functions $\Psi$ and $W$ such that

$$\begin{aligned} \Psi(x,u) &= \Psi(x,0) + \Psi(0,u) + \delta\Psi(x,0)\Psi(0,u), \\ W(p) &= \frac{\Psi\left((x,x) \circ_p (y,y)\right) - \Psi(y,y)}{\Psi(x,x) - \Psi(y,y)} \quad (x > y \geq 0), \end{aligned}$$

and, under some conditions, there is a constant $\gamma > 0$ such that

$$\Psi(x,0) = \gamma\Psi(0,x),$$

which is quite restrictive.

The property characterizing the form of $\Psi(x,u)$ is the Thomsen condition, (23). We showed next that for most people the ears are not symmetric in the sense that $(x,u) \nsim (u,x)$, in which case $\delta = 0$ is equivalent to bisymmetry of the operation

$\oplus_s$. The property underlying the second expression, the one involving $\circ_p$, is proportion commutativity, (25). Axiomatized by themselves these representations really are $\Psi$ and $\Psi_\Theta$, where $\Theta$ is an additive representation over stimulus pairs, and they are not automatically the same function. To establish that equality requires two linking expressions, simple joint presentation decomposition, (31), and one of two forms of segregation, either (32) or (33). These are types of distribution conditions.

Next we took up the form of $\Psi(x,u)$ in terms of the intensities $x$ and $u$. The property of $\sigma-$MI, (34) and (35), turns out to be equivalent to $\Psi$ being a sum of two power functions with the exponents in the ratio of $\sigma$. A predicted linear regression permits one to estimate $\sigma$. We also explored a simple filtering model to allow one to account for the, to us, unexpected phenomena connected with asymmetric matching. If it takes the form of an attenuation factor $\eta$, one can show that none of the tests of properties that we used with asymmetric matching were invalidated by the filtering. We gave formulae for estimating $\eta$ and $\gamma$, respectively (40) and (41).

Our final topic was the form of the ratio estimation predicted by the theory, and that rests heavily on the form of $W(p)$ as a function of $p$. We explored two cases: where ratio estimates are power functions $W(1) \neq 1$, and one where $W(1) = 1$ and $W$ is a Prelec one. Unlike the power function which is either concave or convex, the Prelec function, which includes the power function as a special case, can also be, depending upon the parameter pairs, either S-shaped or inverse S-shaped. Both offered accounts of magnitude methods without a standard and of the ubiquitous sequential effects.

The case when $W$ is a power function leads to a prediction that has not been explored. We described a straightforward experiment aimed at testing the power function but with $W(1) \neq 1$. The alternative explored, the Prelec function for $W$, has been shown to be equivalent to a behavioral property called reduction invariance, (57); two studies, one of them ours, show that this condition fails. Thus the problem of the form of $W$ remains open.

## 7.2 Summary of experimental results

The theory discussed implies that properties 3, 5, 6, and 7 in Table 4 below should hold. Although the results are not perfect, we are reasonably satisfied. Had property 1–symmetry in the sense that $(x,u) \sim (u,x)$–been sustained, which it was not, we could have used a somewhat simpler theoretical development formulated for utility theory. Given that we must deal with the asymmetric case in most instances, we have the representation

$$\Psi(x,u) = \Psi(x,0) + \Psi(0,u) + \delta\Psi(x,0)\Psi(0,u).$$

It is simplified to $\delta = 0$ iff bisymmetry, Property 4, holds, which was well supported. And given that fact, then from multiplicative invariance, Property 2, when it is sustained, two things follow: First, filtering in asymmetric matches does not affect the tests of any of the basic properties. Second, the property of multiplicative invariance is the behavioral equivalent to the psychophysical function being a sum of power functions. We found that roughly half of the respondents satisfied $1-$MI. For the other half we had hoped that using an estimate of $\sigma$ and experimentally checking $\sigma-$MI would improve matters. This did not seem to happen. So additional work on the dependence of $\Psi(x, u)$ on $x$ and on $u$ is clearly needed.

| | Property | #R | #Tests | #Failures |
|---|---|---|---|---|
| 1. | JP-symmetry | 15 | 45 | 23 |
| 2. | Thomsen | 12 | 24 | 5 |
| 3. | Bisymmetry | 6 | 6 | 0 |
| 4. | Prop. Comm. | 4 | 4 | 0 |
| 5. | JP-Decomposition | 4 | 8 | 2 |
| 6. | Segregation | 4 | 10 | 2 |
| 7. | $1-$MI$^*$ | 22 | 44 | 13 |
| 8. | Reduction invariance | 6 | 12 | 12 |

$^*$12 Rs passed both tests

Table 4: Summary of experimental results.

Given the potential for experimental artifacts, we have concluded that sufficient initial support for the theory has been received to warrant further investigation–both for auditory intensity and for other interpretations of the primitives.

A third issue was peculiarities in asymmetric matching which were explained in terms of a filtering model. Should a power function model with exponents in the ratio $\sigma$ obtain, we gave estimation equations for the three free parameters: $\gamma, \sigma, \eta$.

## 7.3 Conclusions

The studies summarized here seem to establish the following points.

- As in classical physics, one does a lot better by having two or more interlocked primitive structures rather than just one in arriving at constrained representations. Our structures were $\langle \mathbb{R}_+ \times \mathbb{R}_+, \succsim \rangle$, which we reduced to the one dimensional structure $\langle \mathbb{R}_+, \geq, \oplus_i \rangle$, and $\langle \mathbb{R}_+ \times \mathbb{R}_+, \succsim, \circ_p \rangle$, which we reduced to the one dimensional structure $\langle \mathbb{R}_+, \geq, \circ_{p,i} \rangle$.
- The adequacy of such a representation theory that has both free functions and free parameters can be judged entirely in terms of parameter-free properties without, at any point, trying to fit the representations themselves to data. Again, this is familiar from classical physics.
- As usual, more needs to be done. Among the most obvious things are:
  - Collect more data. Several specific experiments were mentioned.
  - Continue to try to improve the experimental methodology.
  - Try to extend the theory to encompass auditory frequency as well as intensity.
  - Study interpretations of the primitives other than auditory intensity.

  - Try to find the mathematical forms for $\psi(x)$ and for $W(p)$ are each characterized by a behavioral property that is sustained empirically.

- We are the first to admit, however, that the approach taken is no panacea:

  - We do not have the slightest idea how to axiomatize response times in a comparable fashion.
  - What about probabilistic versions of the theory? Everyone knows that when stimuli are close together, they are not perfectly discriminated and so not really algebraically ordered. Certainly this was true of our data, especially for our data involving ratio productions. Recognition of this fact has, over the years, led to probabilistic versions of various one dimensional ordered structures. But the important goal of blending probabilities with two interacting structures, $\oplus$ and $\circ_p$, in an interesting way has proved to be quite elusive.
  - Also we do not know how to extend the approach to dynamic processes that, at a minimum, seem to underlie both the learning that goes on in psychophysical experiments and the ever-present sequential effects. One thing to recall about dynamic processes in physics is that they are typically formulated in terms of conservation laws (mass, momentum, angular momentum, energy, spin, etc.) that state that certain quantities, definable within the dynamic system, remain invariant over time. Nothing really comparable seems to exist in psychology. Should we be seeking such invariants? We should mention that such invariants always correspond to a form of symmetry. In some systems the symmetry is captured by the set of automorphisms and in others by more general groups of transformations. For further detail see Luce et al. (1990), Narens (2002), and Suppes (2002).

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