Moments of Stopping Times and Scalar Timing

Weber's law for time

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■ 1. Using the Derivatives of Wald's identity. This does not lead to a general solution

■ 2. Use Brownian motion to a single barrier (drift ≥ 0)

We show that for Brownian motion to barrier, the hitting times obey scalar timing (constant CV) if the gaussian increments possess variance proportional to the mean. We do this two ways: using the pdf for the first crossing times and using the moment generating function. The idea is to verify the method for gaussian increments and then generalize to other processes.

■ 2.1 Use the PDF to derive moments for the absorption times (and Weber fraction)

■ 2.2. Use the MGF to derive moments for the absorption times (and Weber fraction)

■ 3. Derive the MGF for absorption times when the diffusion is driven by "difference of Poisson" increments

Before considering the ΔPoisson increments, let's we review the derivation for Brownian motion (Karlin & Taylor, Chapter 7, p. 381ff). This is sort of a practice run.

■ 3.1 Gaussian increments

■ 3.2 Difference of Poissons

Consider two Poisson intensities, α and β. The mgf for these random values are given by

```
M₁ [ θ_] := e^((e^-θ - 1) α
M₂ [ θ_] := e^((e^-θ - 1) β
```

and the difference variable, D, has a distribution whose mgf is

```
M_D[ θ_] := M₁[ θ] M₂[ -θ]
FullSimplify[M_D[ θ]]
```

$$e^{(-1+e^{-\theta})\,\alpha+(-1+e^{\theta})\,\beta}$$

Let X(n) represent discrete Brownian motion, that is the sum of a sequence of n gaussian random numbers. We know that

```
V[n_] := M_D[λ]⁻ⁿ Exp[λ X]
```

is a martingale. We proved this in the Wald_Indentity notebook. I am following K & T here by using λ instead of the traditional θ for the argument of M.

```
V[n]
```

$$e^{X\,\lambda}\left(e^{(-1+e^{-\lambda})\,\alpha+(-1+e^{\lambda})\,\beta}\right)^{-n}$$

Define

```
θ = (-1 + e⁻λ) α + (-1 + eλ) β
```

$$(-1+e^{-\lambda})\,\alpha+(-1+e^{\lambda})\,\beta$$

This yields a simpler experssion for V[n]:

```
eˣ λ e⁻ⁿ θ
```

$$e^{-n\,((-1+e^{-\lambda})\,\alpha+(-1+e^{\lambda})\,\beta)+X\,\lambda}$$

Because V is a martingale, we can apply the optional sampling theorem to the stopped process at X = A and know nonetheless that the expectation is 1. For the stopped process, $e^{X\lambda}$ is not a random variable and we may write

```
Expectation[eˣ λ e⁻ⁿ θ] = 1
eᴬλ Expectation[e⁻ⁿ θ] = 1
Expectation[e⁻ⁿ θ] = eᴬλ
```

We recognize the left side as the Laplace transform of the stopping times. All that is needed is to express λ in terms of θ

```
Unset[θ]
```

```
Solve[ (-1 + e⁻λ) α + (-1 + eλ) β - θ == 0, λ]
```

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found.

$$\left\{\left\{\lambda\to\text{Log}\left[\frac{\alpha+\beta-\sqrt{-4\,\alpha\,\beta+(-\alpha-\beta-\theta)^2}+\theta}{2\,\beta}\right]\right\},\left\{\lambda\to\text{Log}\left[\frac{\alpha+\beta+\sqrt{-4\,\alpha\,\beta+(-\alpha-\beta-\theta)^2}+\theta}{2\,\beta}\right]\right\}\right\}$$

We use the first root for reasons I do not understand. I have discovered that if you use the other root, you get negative 1st moments and other odd results. This gives us an expression for the Laplace transform of the absorption times. (Note, so far, this is the only step for which *Mathematica* proved useful.)

$$\text{Exp}\left[\text{A Log}\left[\frac{\alpha + \beta - \sqrt{-4\,\alpha\,\beta + (-\alpha - \beta - \theta)^2} + \theta}{2\,\beta}\right]\right]$$

$$2^{-A}\left(\frac{\alpha + \beta - \sqrt{-4\,\alpha\,\beta + (-\alpha - \beta - \theta)^2} + \theta}{\beta}\right)^A$$

By changing the sign of θ, we get the moment generating function for the absorption times:

$$M_n[\theta\_] = 2^{-A}\left(\frac{\alpha + \beta - \sqrt{-4\,\alpha\,\beta + (-\alpha - \beta + \theta)^2} - \theta}{\beta}\right)^A$$

$$2^{-A}\left(\frac{\alpha + \beta - \theta - \sqrt{-4\,\alpha\,\beta + (-\alpha - \beta + \theta)^2}}{\beta}\right)^A$$

■ 4. Use the MGF for ΔPoisson increments to derive moments for the absorption times (and Weber fraction)

From section 3, we obtain an expression for the moment generating function of the absroption times at positive barrier, A, given random increments drawn from a difference of Poissons with rates α and β:

$$M[\theta\_, A\_, \alpha\_, \beta\_] := 2^{-A}\left(\frac{\alpha + \beta - \theta - \sqrt{-4\,\alpha\,\beta + (-\alpha - \beta + \theta)^2}}{\beta}\right)^A$$

First derivative with respect to θ

```
D[M[θ, A, α, β], θ]
```

$$\frac{2^{-A}\,A\left(-1 - \frac{-\alpha - \beta + \theta}{\sqrt{-4\,\alpha\,\beta + (-\alpha - \beta + \theta)^2}}\right)\left(\frac{\alpha + \beta - \theta - \sqrt{-4\,\alpha\,\beta + (-\alpha - \beta + \theta)^2}}{\beta}\right)^{-1+A}}{\beta}$$

Use this to define a function that evaluates the 1st derivative (I could not get Mathematica to do this in 1 step).

$$M1F[\theta\_, A\_, \alpha\_, \beta\_] := \frac{2^{-A}\,A\left(-1 - \frac{-\alpha - \beta + \theta}{\sqrt{-4\,\alpha\,\beta + (-\alpha - \beta + \theta)^2}}\right)\left(\frac{\alpha + \beta - \theta - \sqrt{-4\,\alpha\,\beta + (-\alpha - \beta + \theta)^2}}{\beta}\right)^{-1+A}}{\beta}$$

Does this return a reasonable 1st moment? Yes, so long as we use the 1st root.

```
M1F[0, 30, 6, 5]
```

```
30
```

Define a function that returns the 1st moment

```
moment1[A_, α_, β_] := M1F[0, A, α, β]
```

```
moment1[30, 6, 5]
```

30

Second derivative of the MGF with respect to θ

```
D[M[θ, A, α, β], {θ, 2}]
```

$$2^{-A} \left(\frac{(-1+A) \, A \left(-1 - \frac{-\alpha-\beta+\theta}{\sqrt{-4\,\alpha\,\beta+(-\alpha-\beta+\theta)^2}}\right)^2 \left(\frac{\alpha+\beta-\theta-\sqrt{-4\,\alpha\,\beta+(-\alpha-\beta+\theta)^2}}{\beta}\right)^{-2+A}}{\beta^2} + \right.$$

$$\frac{1}{\beta} \left(A \left(\frac{(-\alpha-\beta+\theta)^2}{(-4\,\alpha\,\beta+(-\alpha-\beta+\theta)^2)^{3/2}} - \frac{1}{\sqrt{-4\,\alpha\,\beta+(-\alpha-\beta+\theta)^2}} \right) \right.$$

$$\left. \left. \left(\frac{\alpha+\beta-\theta-\sqrt{-4\,\alpha\,\beta+(-\alpha-\beta+\theta)^2}}{\beta} \right)^{-1+A} \right) \right)$$

Define a function for the 2nd derivative

```
M2F[θ_, A_, α_, β_] :=
```

$$2^{-A} \left(\frac{(-1+A) \, A \left(-1 - \frac{-\alpha-\beta+\theta}{\sqrt{-4\,\alpha\,\beta+(-\alpha-\beta+\theta)^2}}\right)^2 \left(\frac{\alpha+\beta-\theta-\sqrt{-4\,\alpha\,\beta+(-\alpha-\beta+\theta)^2}}{\beta}\right)^{-2+A}}{\beta^2} + \frac{1}{\beta} \left(A \left(\frac{(-\alpha-\beta+\theta)^2}{(-4\,\alpha\,\beta+(-\alpha-\beta+\theta)^2)^{3/2}} - \right.\right.\right.$$

$$\left.\left.\left. \frac{1}{\sqrt{-4\,\alpha\,\beta+(-\alpha-\beta+\theta)^2}} \right) \left(\frac{\alpha+\beta-\theta-\sqrt{-4\,\alpha\,\beta+(-\alpha-\beta+\theta)^2}}{\beta} \right)^{-1+A} \right) \right)$$

```
M2F[0, 30, 6, 5]
```

1230

Define a function that returns the 2nd moment

```
moment2[A_, α_, β_] := M2F[0, A, α, β]
```

The variance is

```
var[A_, α_, β_] := moment2[A, α, β] - moment1[A, α, β]^2
```

```
moment1[30, 6, 5]
moment2[30, 6, 5]
var[30, 6, 5]
```

30

1230

330

Weber fraction

$$\texttt{FullSimplify}\left[\frac{\sqrt{\texttt{M2F}[0,\,A,\,\alpha,\,\beta]-\texttt{M1F}[0,\,A,\,\alpha,\,\beta]^2}}{\texttt{M1F}[0,\,A,\,\alpha,\,\beta]},\,\{A>0,\,\alpha>0,\,\beta>0,\,\alpha>\beta\}\right]$$

$$\frac{\sqrt{\frac{A\,(\alpha+\beta)}{\alpha-\beta}}}{A}$$

This does givse a constant Weber fraction, so long as $\frac{\alpha+\beta}{\alpha-\beta}$ is constant.
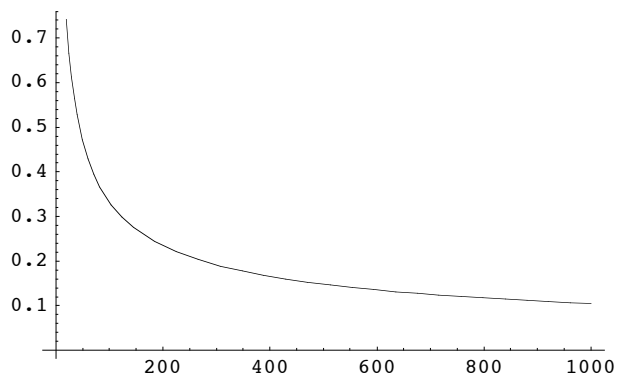
$$\texttt{WeberFraction}[A\_,\,\alpha\_,\,\beta\_]:=\frac{\sqrt{\frac{A\,(\alpha+\beta)}{\alpha-\beta}}}{A}$$

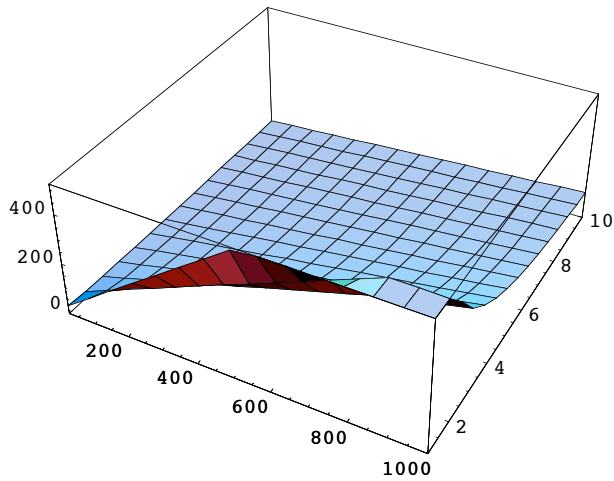`N[WeberFraction[400, 20, 18]]`

0.217945

Here is a plot of the Weber fraction as a function of barrier height for one pair of poissons. Of course the mean time can be scaled by multiplying the pair of rates by a constant.

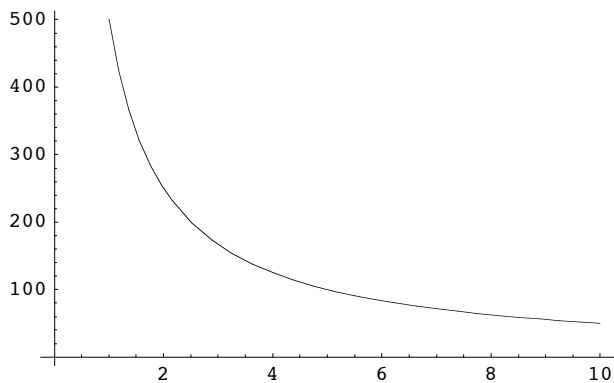`Plot[WeberFraction[A, 6, 5], {A, 20, 1000}]`

- Graphics -

Plot3D[moment1[A, 6 k, 5 k], {A, 20, 1000}, {k, 1, 10}]

- SurfaceGraphics -

Suppose that Poisson rates are computed in 10 msec epochs. The following graph tells us that for k ~3-5, we would time in the half second to 1.5 second range.

Plot[moment1[500, 6 k, 5 k], {k, 1, 10}]

- Graphics -

I like values for k in this range because they might be seen as equivalent to the average of 100 weakly correlated spike trains.This argument will require some work.