Linear Algebra

“Linear algebra has become as basic and as applicable as calculus, and fortunately it is easier”

- Gilbert Strang, *Linear Algebra and its Applications*

**Vectors**

\[ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_N \end{pmatrix} \]

In two or three dimensions, we can draw these as arrows:

- **N=2**
- **N=3**

In higher dimensions, we typically must resort to a “spike-plot”:

- **N=15**
Vector operations

- scalar multiplication
- addition, vector spaces
- length, unit vectors
- inner product (a.k.a. “dot” product)
  - properties: commutative, distributive
  - geometry: cosines, orthogonality test

[on board: geometry]

Vectors as “operators”

- “averager”
- “windowed averager”
- “gaussian averager”
- “local differencer”
- “component selector”

[answers on board]
Inner product with a unit vector

- projection
- distance
- change of coordinates

(on board: geometry)

Linear System

$S$ is a linear system if (and only if) it obeys the principal of superposition:

$$S(a\vec{x} + b\vec{y}) = aS(\vec{x}) + bS(\vec{y})$$

For any input vectors $\{\vec{x}, \vec{y}\}$, and any scalars $\{a, b\}$, the two diagrams at the right must produce the same response:
Linear Systems

- Very well understood (150+ years of effort)
- Excellent design/characterization toolbox
- An idealization (they do not exist!)

- Useful nevertheless:
  - conceptualize fundamental issues
  - provide baseline performance
  - good starting point for more complex models

Implications of Linearity

Input
\( \vec{v} \)

\[ \rightarrow \]

\( L \)

\[ \rightarrow \]

Output
Implications of Linearity

Response to *any* input can be predicted from responses to impulses. This defines the operation of *matrix multiplication*.
Matrix multiplication

- input perspective: weighted sum of columns (from diagram on previous slide)
- output perspective: inner product with rows
- distributive (comes from linearity)
- associative - cascade of two linear systems defines the product of two matrices
- generally not commutative

Matrix multiplication: dimensional consistency
Matrix types

Orthogonal matrices
- square shape (dimensionality-preserving)
- rows are orthogonal unit vectors
- columns are orthogonal unit vectors
- performs a rotation of the vector space (with possible axis inversion)
- preserve vector lengths and angles
- inverse is transpose

Identity matrix

Diagonal matrices
- arbitrary rectangular shape
- all off-diagonal entries are zero
- squeeze/stretch along standard axes
- if non-square: creates/discards axes
- inverse is diagonal, with inverse of non-zero diagonal entries of original

Singular Value Decomposition (SVD)

- $M = U S V^T$, “rotate, stretch, rotate”
- $V$ is input coordinate system (U, output)
- interpretation as sum of outer products
- non-uniqueness (permutations, sign flips)
- nullspace and rangespace
- inverse and pseudo-inverse

[details on board]
\[ M = U S V^T \]