Mathematical Tools
for Neural and Cognitive Science

Fall semester, 2023

Section 1: Linear Algebra

Linear Algebra

“Linear algebra has become as basic and as applicable as calculus, and fortunately it is easier”

- Gilbert Strang, Linear Algebra and its Applications

…and this is even more true today than when the book was published!

Vectors
Ordered lists of numbers, depicted in 3 ways:

In two or three dimensions, we can draw these as arrows:

\[
\vec{f} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\]

N=2

N=3

In higher dimensions, we typically must resort to a “spike-plot”

N=15

\[ x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6 \ x_7 \ x_8 \ x_9 \ x_{10} \ x_{11} \ x_{12} \ x_{13} \ x_{14} \ x_{15} \]
Vector operations

- scalar multiplication
- addition, vector spaces
- length, unit vectors
- inner product (a.k.a. “dot” product)
  - definition/notation: sum of pairwise products
  - geometry: cosines, squared length, orthogonality test

(on board: geometry)

Inner product with a unit vector

- projection onto line
- distance to line/plane
- change of coordinates

(on board: geometry)

Vectors as “operators”

- “averager”
- “windowed averager”
- “smooth averager”
- “local differencer”
- “component selector”

(on board)
Linear System

$S$ is a linear system if (and only if) it obeys the principle of superposition:

$$S(a\vec{x} + b\vec{y}) = aS(\vec{x}) + bS(\vec{y})$$

For any input vectors $\{\vec{x}, \vec{y}\}$ and any scalars $\{a, b\}$, the two diagrams at the right must produce the same response.

Linear Systems

- Very well understood (150+ years of effort)
- Excellent design/characterization toolbox
- An idealization (they do not exist!)
- Useful nevertheless:
  - conceptualize fundamental issues
  - provide baseline performance
  - provide building blocks for more complex models

Implications of Linearity

- Input
- Output
Implications of Linearity

write input vector as weighted sum of "impulse vectors" "axis vectors" "standard basis"

Response to any input can be computed from responses to impulses
This defines the operation of matrix multiplication

Matrix multiplication
Two interpretations of $M \vec{v}$

input perspective: weighted sum of columns
output perspective: inner product with rows

System response to first axis, $e_1$
Matrix multiplication

- two interpretations of $M\vec{v}$:
  - weighted sum of columns
  - inner products with rows
- transpose $A^T$, symmetric matrices ($A = A^T$)
- distributive property: directly from linearity!
- associative property: cascade of two linear systems is linear. Defines matrix multiplication.

Cascaded linear systems $\Rightarrow$ product of matrices
Matrix multiplication

- two interpretations of $M\vec{v}$:
  - “input perspective”: weighted sum of columns
  - “output perspective”: inner product with rows
- transpose $A^T$, symmetric matrices ($A = A^T$)
- distributive property: directly from linearity!
- associative property: cascade of two linear systems is linear. Defines matrix multiplication.
- generally not commutative ($AB \neq BA$), but note that $(AB)^T = B^T A^T$
- vectors as matrices: Inner products, Outer products

Orthogonal matrices

- square shape (dimensionality-preserving)
- rows are orthogonal unit vectors
- columns are orthogonal unit vectors
- performs a rotation of the vector space (with possible axis inversion)
- preserve vector lengths and angles (and thus, dot products)
- inverse is transpose

Diagonal matrices

- arbitrary rectangular shape
- all off-diagonal entries are zero
- squeeze/stretch along standard axes
- if non-square, creates/discards axes
- inverse is diagonal, with inverse of non-zero diagonal entries of original

All matrices

Singular Value Decomposition (SVD)

Any matrix $M$ can be factorized as

$$M = U S V^T$$

with $U$, $V$ orthogonal, $S$ diagonal

- geometry: “rotate, stretch, rotate”
- columns of $V$ are basis for input coordinate system
- columns of $U$ are basis for output coordinate system
- $S$ rescales axes, and determines what “gets through”
SVD geometry (in 2D)

Apply $M$ to four vectors (heads at colored points):

$$M = USV^T$$

(note order of transformations)

Singular Value Decomposition (SVD)

Any matrix $M$ can be factorized as

$$M = USV^T$$

with $U, V$ orthogonal, $S$ diagonal

- unique, up to permutations and sign flips
- sum of “outer products”
- nullspace and rangespace
- inverse and pseudo-inverse

[details on board]

$$M\hat{x} = \sum_k \hat{u}_k \left( s_k (\hat{v}_k^T \hat{x}) \right) = \sum_k s_k \left( \hat{u}_k \hat{v}_k^T \right) \hat{x} \quad \text{(sum of outer products)}$$

Any matrix $M$ can be factorized as

$$M = USV^T$$

with $U, V$ orthogonal, $S$ diagonal

- unique, up to permutations and sign flips
- sum of “outer products”
- nullspace and rangespace
- inverse and pseudo-inverse

[details on board]
The SVD (Singular Value Decomposition) of a matrix $M$ is given by $M = U S V^T$, where $U$ and $V$ are orthogonal matrices, and $S$ is a diagonal matrix containing the singular values of $M$.

- $U$ represents the orthogonal basis for the range space of $M$.
- $V$ represents the orthogonal basis for the null space of $M$.
- $S$ contains the singular values of $M$.

In the case of a matrix with a null space (all zeros), the null space basis is represented by all zeros, and the range space basis is represented by the non-zero singular values.