Section 1: Linear Algebra

“Linear algebra has become as basic and as applicable as calculus, and fortunately it is easier”
- Gilbert Strang, *Linear Algebra and its Applications*
Vectors

Vector operations

- scalar multiplication
- addition, vector spaces
- length, unit vectors
- inner product (a.k.a. “dot” product)
  - properties: commutative, distributive
  - geometry: cosines, orthogonality test

(on board: geometry)
Vectors as “operators”

- “averager”
- “windowed averager”
- “gaussian averager”
- “local differencer”
- “component selector”

[on board]

Inner product with a unit vector

- projection
- distance to line
- change of coordinates

[on board: geometry]
Linear System

$S$ is a linear system if (and only if) it obeys the principle of superposition:

$$S(ax + by) = aS(x) + bS(y)$$

For any input vectors $\{x, y\}$, and any scalars $\{a, b\}$, the two diagrams at the right must produce the same response:

Linear Systems

- Very well understood (150+ years of effort)
- Excellent design/characterization toolbox
- An idealization (they do not exist!)
- Useful nevertheless:
  - conceptualize fundamental issues
  - provide baseline performance
  - good starting point for more complex models
Implications of Linearity

Input

Output

"impulse" vectors
"standard basis"
"axis vectors"
Implications of Linearity

Response to any input can be predicted from responses to impulses. This defines the operation of matrix multiplication.

Matrix multiplication

- Two interpretations of $M\vec{v}$ (see next slide):
  - input perspective: weighted sum of columns
    (from diagram on previous slide)
  - output perspective: inner product with rows
- transpose $A^T$, symmetric matrices ($A = A^T$)
- distributive property (directly from linearity!)
- associative property - cascade of two linear systems defines the product of two matrices
- generally not commutative ($AB \neq BA$), but note that $(AB)^T = B^TA^T$
input perspective: weighted sum of columns

\[ M \quad \vec{v} \]

output perspective: dot product with rows

\[ M \quad \vec{v} \]

Matrix multiplication: dimensional consistency

\[
\begin{bmatrix}
& & \\
& & \\
& & \\
& & \\
& & \\
n & & \\
& & \\
& & \\
& & \\
& & \\
n & & \\
& & \\
& & \\
& & \\
& & \\
\end{bmatrix}
\begin{bmatrix} n \end{bmatrix} =
\begin{bmatrix}
& & \\
& & \\
& & \\
& & \\
& & \\
n & & \\
& & \\
& & \\
& & \\
& & \\
n & & \\
& & \\
& & \\
& & \\
& & \\
\end{bmatrix}
Orthogonal matrices
- square shape (dimensionality-preserving)
- rows are orthogonal unit vectors
- columns are orthogonal unit vectors
- performs a rotation of the vector space (with possible axis inversion)
- preserve vector lengths and angles (and thus, dot products)
- inverse is transpose

Diagonal matrices
- arbitrary rectangular shape
- all off-diagonal entries are zero
- squeeze/stretch along standard axes
- if non-square, creates/discards axes
- inverse is diagonal, with inverse of non-zero diagonal entries of original

Identity matrix

Singular Value Decomposition (SVD)
- can express any matrix as $M = U S V^T$
- “rotate, stretch, rotate”
  - columns of $V$ are basis for input coordinate system
  - columns of $U$ are basis for output coordinate system
  - $S$ rescales axes, and determines what gets through
- interpretation as sum of “outer products”
- non-uniqueness (permutations, sign flips)
- nullspace and rangespace
- inverse and pseudo-inverse

(details on board)
SVD geometry (in 2D)
Consider applying $M$ to four vectors (colored points)

$$M = U S V^T$$

(note order of transformations!)

$$M \bar{w} = \sum_k s_k (\bar{v}_k^T \bar{w}) \bar{u}_k = \sum_k s_k (\bar{u}_k \bar{v}_k^T) \bar{w}$$

"singular values"