Section 3a: Early auditory system
(an extended LSI example)
Figures 11.18
The movements of the basilar membrane's vibration at different frequencies ranging from 25 to 10,000 Hz, as measured by Békésy (1940).
Figure 6-14 Threshold response curves for auditory nerve fibers in the cat (based on Whitfield, 1968).
Critical Bands

- Loudness summation
- Critical band masking

“Designing” Filters

\[
\begin{align*}
y(t) &= V_C = \frac{1}{C} \int dt \\
x(t) - y(t) &= V_R = IR \\
x - y &= RC \frac{dy}{dt} = \tau \frac{dy}{dt}
\end{align*}
\]

For \( x = 0 \), solutions have the form \( y = Ae^{-\frac{t}{\tau}} \)

**Frequency response of RC low-pass filter**: log-log Bode plot

\[
\tilde{y}(\omega) = \frac{1}{i\omega + 1}
\]
Cascaded or Gamma Filters

\[ y(n) \propto t^{n-1} e^{-t/\tau} \]

\[ \tilde{y}(\omega) = \left( \frac{1}{i\tau\omega + 1} \right)^n \]

Gammatone Filters

\[ y \propto t^{n-1} e^{-t/\tau} \cos(\omega t - \phi) \]

Speech Spectrogram
Example – Bass/Treble filters

Miles Davis
“Half Nelson”

Bass only:  
Treble only:  

Time

Signal

Filter

Frequency content

Low-pass or Bass Filter

High-pass or Treble Filter
Example – Bass/Treble filters

Miles Davis
"Half Nelson"

Bass only:

Treble only:

Time

Filter

Frequency content

Low-pass or Bass Filter

High-pass or Treble Filter

Example – Bass/Treble filters

Miles Davis
"Half Nelson"

Bass only:

Treble only:

Time

Filter

Frequency content

Low-pass or Bass Filter

High-pass or Treble Filter

Fourier spectrum representation of sound

Period

Pressure

Time

Frequency

Amplitude

Pressure

Time

Frequency

Amplitude
This works for all “musical sounds”

Fundamental frequency and harmonics

Standing waves in a vibrating string
Flute (open pipe) harmonics

Other notes (shorten the pipe)

Reinforcing a harmonic (forcing a “node”)

Musical temperament

• How should you tune your piano, i.e., what “fundamental” frequency should be associated with each note?
• Equivalently: Where should the guitar frets be placed and strings be tuned? Where should the air holes on the flute be cut?
• http://virtualpiano.net
Musical basics

• The western scale has 12 notes: C, C#, D, D#, E, F, F#, G, G#, A, A#, B

• Half steps up from one note are the same ("enharmonic") as half steps down from a higher note. For example, C# = D♭ ("C sharp equals D flat")

• Perception of pitch is "circular", so that after B, then next higher note is C yet again, described as one "octave" above the next lower note named C

• Perception of pitch depends on ratios. In particular, an increase of one octave, such as from one C to the next higher C, DOUBLES the frequency (C = 256 Hz, next C = 512 Hz).

Musical harmonics

• An instrument playing C (frequency \( f = 256 \text{ Hz} \)) will generally also produce harmonics \( 2f, 3f, 4f, ... \)

• What notes do these harmonics sound like when heard on their own?

  Sound like C
  
  Sound like E
  
  Sound like G
  
  Sounds like B♭

Aside: Beat frequencies

• What happens when unison notes are mistuned?

\[
\cos \left( 2\pi \left( f - \frac{\Delta f}{2} \right) t \right) + \cos \left( 2\pi \left( f + \frac{\Delta f}{2} \right) t \right) = 2 \cos \left( 2\pi ft \right) \cos \left( 2\pi \Delta ft \right)
\]

• That is, you hear the average frequency "beating", i.e., modulating up and down in volume, at the "difference" frequency (called "interference" in physics): 256 Hz + 258 Hz
Musical harmonics

Thus, the most “consonant” (pleasant sounding, as opposed to “dissonant”) interval other than the octave (which sounds like unison) is the “perfect fifth” from C to G (seven half-steps), with a frequency ratio of 3 (or 3/2, 3/4, 6, 12, ...), because then when C & G are played together, the harmonics don't “beat”.

Musical temperament, contd.

- A tuning system is a choice of what frequencies (or frequency ratios) correspond to the notes: C, C#, D, D#, E, F, F#, G, G#, A, A#, B.
- A “just” temperament tries, when possible, to have the frequency ratios be simple fractions, such as 3/2 for G relative to C.

The circle of 5ths

- C
- F
- B♭
- E♭
- A♭
- D♭
- G♭
Musical temperament, contd.

- Thus, a just temperament, based on fractions, cannot keep the intended “perfect” interval (e.g., 3/2 for the interval of a 5th) in every key.
- This leads to keys with so-called wolf intervals, making them unusable.
- There are many different just temperaments.
- By Bach’s time, Western music did not use Pythagorean tuning, but rather something called “quarter-comma meantone”, which is based on making most of the Major 3rds have a ratio of 5/4, but also has several wolf intervals.
- How can you split the octave (12 semitones) equally?
Equal temperament

This works (mathematically, if not musically)!

Comparing temperaments

- Tuning systems are based on frequency ratios.
- Ratios are best described logarithmically, because of the property that log turns multiplication into addition:
  \[ \log(ab) = \log(a) + \log(b) \]
  so that going up by two musical intervals (a product of two ratios) looks like addition.
- The usual unit used for this is the "cent", with 100 cents per chromatic step and 1200 cents per octave, so that an interval from \( f_1 \) to \( f_2 \), in cents, is \( 1200 \log_2 \left( \frac{f_2}{f_1} \right) \)
Comparing temperaments

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What does this have to do with Bach?

The Well-Tempered Clavier (first book: 1722) contains one prelude and one fugue in each of the 12 Major and minor keys.

The point is: In equal temperament, and in well-designed (i.e., "well tempered") unequal temperaments, you can play all 24 pieces in a row without retuning your clavier.

The fact that Bach used the term "well tempered" almost certainly means he was not using equal temperament at the time. The term comes from the writings of a contemporary, Werckmeister, and likely he was using a tuning suggested by Werckmeister.

The subtleties

Bach's prelude (BWV 998) for lute or clavier:

As I said, Bach likely was NOT using equal temperament but, rather, an unequal temperament from a contemporary (a system called Werckmeister III) that allows one to play in all keys, but gives each key a different character. Here is Bach's Chromatic Fantasy for harpsichord:

Equal temperament (Christophe Rousset)
Werckmeister III (Rebecca Pechefsky)
The subtleties

Bach’s prelude (BWV 998) for lute or clavier:

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- Equal temperament (Christophe Rousset)
- Werckmeister III (Rebecca Pechefsky)

The Well-Tempered Clavier

Thus, The Well-Tempered Clavier (Book One: 1722; Book Two: 1742) are two books of 24 preludes and 24 fugues, one pair in each of the 12 major and 12 minor keys. It effectively serves as an advertisement for "well temperament" (not likely equal temperament, and research suggests he used Werckmeister III).

Here is Rebecca Pechefsky performing Book One, Prelude No. 1 in C Major, on harpsichord, tuned in Werckmeister III.