“Independent” Components Analysis (ICA)

For Linearly Transformed Factorial (LTF) sources: guaranteed independence
(with some minor caveats)

[Comon 94; Cardoso 96; Bell/Sejnowski 97; ...]

Model II (LTF)

Coefficient density:

<table>
<thead>
<tr>
<th>Coefficient Density</th>
<th>Basis Set</th>
<th>Image</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Coefficient Density" /></td>
<td><img src="image2.png" alt="Basis Set" /></td>
<td><img src="image3.png" alt="Image" /></td>
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<td><img src="image4.png" alt="Coefficient Density" /></td>
<td><img src="image5.png" alt="Basis Set" /></td>
<td><img src="image6.png" alt="Image" /></td>
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<td><img src="image7.png" alt="Coefficient Density" /></td>
<td><img src="image8.png" alt="Basis Set" /></td>
<td><img src="image9.png" alt="Image" /></td>
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Trouble in paradise

• Statistics: Images don’t obey ICA source model
  - Image subband coefficients are clearly not independent samples (by visual inspection!)
  - The responses of ICA filters are highly dependent [Wegmann & Zetzsche 90, Simoncelli 97]
  - All bandpass filters give sparse marginals [Baddeley 96] \(\Rightarrow\) Oriented filters are a shallow optimum [Bethge 06; Lyu & Simoncelli 08]

• Biology: Visual system uses a cascade
  - Where’s the retina? The LGN?
  - What happens after V1? Why don’t responses get sparser? [Baddeley etal 97; Chechik etal 06]
Indications that the model is weak...

Sample from model

Image, ICA-transformed and Gaussianized

- Large-magnitude subband coefficients are found at neighboring positions, orientations, and scales.
Conditional densities reveal nonlinear dependencies

[Simoncelli 97; Buccigrossi&Simoncelli 99; Wainwright&Simoncelli 99; Schwartz&Simoncelli 01]

Joint densities

- Nearby: densities are approximately circular/elliptical
- Distant: densities are approximately factorial

[Simoncelli, ‘97; Wainwright&Simoncelli, ‘99]
Non-Gaussian elliptical observations and models of natural images:

- Zetzsche & Krieger, 1999;
- Huang & Mumford, 1999;
- Wainwright & Simoncelli, 1999;
- Hyvärinen and Hoyer, 2000;
- Parra et al., 2001;
- Srivastava et al., 2002;
- Sendur & Selesnick, 2002;
- Teh et al., 2003;
- Gehler and Welling, 2006
- Lyu & Simoncelli, 2008
- etc.

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**Spherical vs Sparse**

- Histograms, kurtosis of projections of image blocks onto random unit-norm basis functions.
- These imply data are closer to spherical than factorial

[Lyu & Simoncelli 08]
[Lyu & Simoncelli, 2008]
[Lyu & Simoncelli, 2008]
Putting it all together...

- Subband coefficients are marginally non-Gaussian
- Coefficient pairs, or local clusters, are approximately elliptical
- Image subbands contain a small number of very large coefficients (that’s what lets us separate them from noise), and these tend to occur near each other
- So suppose coefficients are locally Gaussian, but the variance is fluctuating over the image (known as heteroscedasticity)

Marginal statistics - sound

Signal is *heteroskedastic* (has varying variance):

\[ p(x) \neq p(x) \]
Modeling heteroscedasticity

Assume a hidden scaling variable for each patch

Gaussian scale mixture (GSM) [Andrews & Mallows 74]:
\[ \tilde{x} = \sqrt{z} \tilde{u} \]

- \( \tilde{u} \) is Gaussian, \( z > 0 \)
- \( z \) and \( \tilde{u} \) are independent
- \( \tilde{x} \) is elliptically symmetric, with covariance \( \propto C_u \)
- marginals of \( \tilde{x} \) are leptokurtotic

[Wainwright & Simoncelli 99]
Denoising: Joint

\[ \mathbb{E}(x|\tilde{y}) = \int dz \ P(z|\tilde{y}) \mathbb{E}(x|\tilde{y}, z) \]
\[ = \int dz \ P(z|\tilde{y}) \left[ zC_u(zC_u + C_w)^{-1}\tilde{y}\right]_{\text{ctr}} \]

where

\[ P(z|\tilde{y}) = \frac{P(\tilde{y}|z) \ P(z)}{P(\tilde{y})}, \quad P(\tilde{y}|z) = \frac{\exp(-\tilde{y}^T(zC_u + C_w)^{-1}\tilde{y}/2)}{\sqrt{(2\pi)^N|zC_u + C_w|}} \]

Numerical computation of solution is reasonably efficient if one jointly diagonalizes \( C_u \) and \( C_w \) ...}

[Portilla et. al. 2003]

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Example estimators

Estimators for the scalar and single-neighbor cases

[Portilla et. al. 2003]
Original

Noisy
(8.1 dB)

UndWvlt
Thresh
(19.0 dB)

BLS-GSM
(21.2 dB)

[Portilla et. al. 2003]

noisy
(4.8)

I-linear
(10.61)

II-marginal
(11.98)

III-GSM
nbd: $5 \times 5 + p$
(13.60)

[Portilla et. al. 2003]
Real sensor noise

400 ISO

GSM denoised

[Portilla et. al. 2003]