
Synaptic depression is a form of short-term plasticity in which a neuron’s synaptic efficacy is reduced abruptly just after firing of a spike.

The synaptic efficacy variable decays back towards its normal value in between spikes.

When an action potential arrives at a synaptic terminal, the efficacy of the synapse is determined by the value of the synaptic efficacy variable just before spike arrival.

The synaptic efficacy variable either controls the probability of synaptic transmission or the magnitude of the unitary postsynaptic event.
Simple model for synaptic depression

$S$ - synaptic efficacy variable
(synaptic resources)

$0 < S < 1$

Every time a neuron fires, $S$ is reduced instantaneously by a fixed fraction, and $S$ grows with an exponential time course towards 1 between firings.

Define $S^-$ as the value just before firing and $S^+$ as the value just after:

$S^+ = f S^-$, where $0 < f < 1$

In differential equation form for a firing event at $t = 0$

\[
\frac{dS}{dt} = \frac{(1-S) - (\ln(f))\delta(t)}{T}
\]
To find \( S(0^+) - S(0^-) \), divide by \( S \) and integrate from \( t=0^+ \) to \( t=0^- \):

\[
\frac{1}{S} \frac{dS}{dt} = \frac{1-S}{T} + (\ln f) \delta(t) \quad (2)
\]

\[
\int_{S(0^-)}^{S(0^+)} \frac{1}{S} dS = \int_{0^-}^{0^+} dt \left\{ \frac{1-S}{T} + (\ln f) \delta(t) \right\} \quad (3)
\]

\[
\ln \frac{S(0^+)}{S(0^-)} = \ln f
\]

\[
\frac{S(0^+)}{S(0^-)} = f
\]

More generally for firings occurring at random times \( T_k \):

\[
\frac{dS}{dt} = \frac{1}{T} (1-S) + (\ln f) \sum_{k=1}^{\infty} \delta(t-T_k) \quad (4)
\]

Assume \( T_k \) are governed by a modulated Poisson process.
If firing rate is \( V(t) \) then

\[
\Pr \left( \text{Neuron fires in } (t, t+\Delta t) \right) = V(t) \Delta t + O((\Delta t)^2)
\]  

(6)

Non-overlapping time intervals are independent, so behavior is independent of past history.

Crude approximation at best.

If clock is started at \( t = 0 \), then

\[
\Pr \left( \text{Neuron fires in } (t, t+\Delta t) \right) = V(t) \int_0^t V(t') \, dt' \Delta t
\]  

(7)

If \( T \) = random waiting time to first firing

\[
\frac{f(t)}{T} = V(t) e^{-\int_0^t V(t') \, dt'}
\]  

(8)

is the p.d.f. for \( T \).
Because the clock starts anew after each firing, 

\[ \mathbb{P}(T_k \mid T_{k-1}) = \int_0^T \mathbb{P}(V(t) \leq \epsilon_k) \, dt, \]

is the conditional p.d.f.

In the special case of constant \( V \),

\[ f(t) = V e^{-\lambda t} \quad (10) \]

is the p.d.f. for the random interval \( T \) between firings.

\[ \mu_T = \frac{1}{V} \quad \text{in this case.} \quad (11) \]

mean waiting time
Use of synaptic depression in firing rate models requires some analysis.

It can be shown (see below) that for a population of uncorrelated, statistically equivalent neurons, the population mean of $S$, call it $\bar{S}(t)$

or simply $\bar{S}$ for short, evolves according to

$$\frac{dS}{dt} = \frac{1}{T} (1-S) - \bar{v}(t) (1-f) S \quad \text{(4b.1)}$$

where $\bar{v}(t)$ is the population firing rate.

In some models where $s$ controls the probability of synaptic transmission and where that probability is equal to $\bar{u}$ for $s=1$, the mean probability $p$ is given by

$$p = \bar{u} s \quad \text{(4b.2)}$$
For sake of simplicity, set $u = 1$

The synaptic drive or synaptic output of a synapse is determined by the product of $s$ and the firing rate $v$.

Define

$$y(t) = \overline{g} s(t) v(t) \quad (4c.1)$$

as the synaptic drive.

In a model with conductance driven synapses in the postsynaptic neuron, the excitatory postsynaptic conductance might evolve according to

$$\frac{d}{dt} g_e = -\frac{1}{T_e} g_e + \overline{g} s(t) v(t) \quad (4c.2)$$

(The case in which the unitary postsynaptic conductance event has a single exponential time course.)
Steady-state $s-V$ relationship:

Set $\frac{ds}{dt} = 0 \cdot v(t) = V_0$; solve for $s_0$

\[
    s_0 = \frac{1}{V_0 \tau (1-f) + 1}
\]

(4d.1)

For example, for $\tau = 200 \text{ ms}$, $f = 0.5$

\[
    s_0 \approx 0.1 \text{ when } V_0 = 100 \text{ Hz}
\]

Steady state synaptic drive, $\gamma - \delta$ relationship:

\[
    \delta_0 = \delta \frac{V_0}{1 + V_0 \tau (1-f)}
\]

(4d.2)

Note:
\[
    \delta'_0(V_0) > 0
\]
\[
    \delta''_0(V_0) < 0
\]
Kinetics of s evolution for step changes in V:

\[ \frac{\tau}{1 + V_0 \tau (1-f)} \]
Temporal frequency transfer function for the \( V \rightarrow Y \) transduction:

Let \( V(t) = V_0 + \varepsilon V_1(t) \).

Then \( Y(t) = Y_0 + \varepsilon Y_1(t) + O(\varepsilon^2) \).

Specialize to \( V_1(t) = \text{Re} e^{i\omega t} \).

Then \( Y_1(t) = \text{Re} \left( \frac{\Gamma(w)}{V_0} e^{i\omega t} \right) \).

\[
\Gamma(w) = \frac{\bar{g}}{V_0} \frac{1 + i\omega \tau}{1 + V_0 \tau(1-f)} \frac{1 + i\omega \tau}{1 + V_0 \tau(1-f) + i\omega \tau}
\]

(4.f.1)

The magnitude of \( \frac{\Gamma(w)}{V_0} \) is the gain for the \( V \rightarrow Y \) transduction. \( \frac{\Gamma(w)}{V_0} \) can also be written as

\[
\frac{\Gamma(w)}{V_0} = \frac{\bar{g}}{[1 + V_0 \tau(1-f)]^2} \frac{1 + i\omega \tau}{1 + i\omega \tau} \left( \frac{1}{1 + V_0 \tau(1-f)} \right)
\]
49.

A family of $\Gamma(w)$, temporal frequency transfer functions for different values of $V_0$ (operating points):

\[ \log \left| \frac{\Gamma}{\Gamma_{V=0}} \right| \begin{array}{c} 1 \end{array} \begin{array}{c} 0.1 \end{array} \begin{array}{c} 0.01 \end{array} \begin{array}{c} \log w \end{array} \]

- Low frequencies are filtered out and to a greater extent for higher $V_0$.
-4h-

- Superposition of two input stimuli that give firing rate responses

\[ L^a = L_0 + \varepsilon L^a \] average and modulated components

\[ L^b = L_0 + \varepsilon L^b \]

If \( L^a_{1,0} \) are periodic in particular, but more generally when one attempts to measure the modulated component of the response of order \( \varepsilon \), the relevant approximations for the firing rates are

\[ L^a \rightarrow V^a(t) = V^a_0(\varepsilon^2) + \varepsilon V^a_{1,1}(t) \]

\[ L^b \rightarrow V^b(t) = V^b_0(\varepsilon^2) + \varepsilon V^b_{1,1}(t) \]

\[ L = L_0 + \varepsilon L^a_{1,1} + \varepsilon L^b_{1,1} \]

\[ \rightarrow V(t) \overset{\sim}{=} V_0^{a+b}(\varepsilon^2) + \varepsilon V^a_{1,1}(t) + \varepsilon V^b_{1,1}(t) \]

where it can be

\[ V_0^{a+b}(\varepsilon^2) > V_0^a(\varepsilon^2) \]

\[ V_0^{a+b}(\varepsilon^2) > V_0^b(\varepsilon^2) \]
Consider a population of statistically equivalent neurons. The Poisson description applies to each.

We are concerned only with the random state variable for the neurons

\[ \int_{s}^{b} \rho(s,t) \, ds = \Pr(\text{Neuron drawn at random from the population will have } s \text{ in the interval } (a,b)) \]  

(12)

Consider how this probability changes over time.
Neurons can enter and leave the interval from the evolution of \( s \) towards 1 with an exponential time course. This is a drift contribution.

\[ J_d(a, t) \rightarrow J_d(b, t) \]

Let \( J_d(a, t) \) be the rate at which neurons enter the interval by crossing \( a \).

\( J_d(b, t) \), the rate at which neurons leave by crossing \( b \).

\( J_d \) is a probability flux.
A neuron with $S = s$ moves in the state space with velocity

\[ \frac{dS}{dt} = (1 - S) \tag{13} \]

\[ J_d(a, t) \Delta t \approx \rho(a, t) \Delta S \tag{14} \]

Distance travelled in time $\Delta t$

\[ J_d(a, t) \approx \rho(a, t) \frac{dS}{dt} \Delta t \tag{15} \]

In limit $\Delta t \to 0$

\[ J_d(a, t) = \rho(a, t) \frac{dS}{dt} \bigg|_{S=a} \tag{16} \]

and

\[ J_d(b, t) = \rho(b, t) \frac{dS}{dt} \bigg|_{S=b} \tag{17} \]
Neurons can enter and leave interval through jumps in $S$

$J_f(a,t)$

Any neuron in this $S$ interval will have its $s$ value kicked across $a$ from above if it fires. The probability per unit time of firing is $v(t)$

$J_f(a,t) \equiv$ rate of crossing $a$ by firing
\[ J_f(a,t) = -V(t) \int_a^{\min(\frac{a}{2},1)} \rho(s',t) \, ds' \] (18)

Total flux or (signed) rate of crossing \( s \), \( J \)

\[ J(s,t) = J_d(s,t) + J_f(s,t) \]

\[ = \left( 1-s \right) \rho(s,t) - V(t) \int_s^{\min(s,1)} \rho(s',t) \, ds' \] (19)

\[ \frac{\partial}{\partial t} \int_a^b \rho(s,t) \, ds = J(a,t) - J(b,t) \] (20)

Differentiate with respect to \( b \)

\[ \frac{\partial}{\partial t} \rho(b,t) = -\frac{\partial}{\partial b} J(b,t) \] (21)

\( b \) is any arbitrary \( s \)
\[ \frac{\partial}{\partial t} \rho(s,t) = -\frac{\partial}{\partial s} J(s,t) \quad (22) \]

\[ \frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial s} \left\{ \frac{(1-s)\rho(s,t)}{\min(s,1)} - \nu(t) \int_{s}^{\min(s,1)} \rho(s',t) ds' \right\} \quad (23) \]

We wish to derive an evolution equation for the population mean value of \( S \), \( \mu_s(t) \), where

\[ \mu_s(t) = \frac{1}{S} \int_{0}^{S} sp(s,t) ds \quad (24) \]
Multiply evolution eqn. for \( p \) by \( s \) and integrate:

\[
\int_0^s \frac{\partial p}{\partial t} \, ds = \int_0^s \left( -\frac{2J(s,t)}{\partial s} \right) \, ds
\]

(25)

\[
\frac{1}{s} \int_0^s p(s,t) \, ds = -\int_0^1 s \, \frac{2J}{\partial s} \, ds
\]

(26)

\[
\frac{d}{dt} \frac{M(s,t)}{s} = -\int_0^s s \, \frac{2J}{\partial s} \, ds
\]

(27)

\[
\frac{d}{dt} \frac{M(s,t)}{s} = -\left\{ \left. \frac{1}{s} J(s,t) \right|_0^1 - \int_0^1 J(s,t) \, ds \right\}
\]

(28)

There is no flux across \( 1 \).

\[
\frac{d}{dt} \frac{M(s,t)}{s} = \int_0^1 J(s,t) \, ds
\]

(29)
Consider the integrals of $J_d$ and $J_f$ separately.

\[
\int_{0}^{1} J_d(s,t) \, ds = \int_{0}^{1} \frac{(1-s)\rho(s,t)}{\tau} \, ds = \frac{1}{\tau} \left(1 - \mu_s(t)\right) \quad (30)
\]

\[
\int_{0}^{1} J_f(s,t) \, ds = -V(t) \int_{0}^{\min(\frac{s}{d}, 1)} ds \int_{s}^{\min(\frac{s}{d}, 1)} \rho(s',t) \, ds' \quad (31)
\]

\[
\equiv I
\]

\[
I = \int_{0}^{f} ds \int_{0}^{\frac{s}{d}} \rho(s',t) \, ds' + \int_{0}^{1} ds \int_{s}^{1} \rho(s',t) \, ds' \quad (32)
\]

Call the first double integral $I_1$, and the second $I_2$. 
The trick is to get $s$ out of the limits of integration. This will allow useful switching of the order of integration.

For example, consider

$$I_2 = \int_{s}^{1} \int_{s}^{1} \rho(s', t) \, ds' \, ds \quad (33)$$
\[ I_2 = \int_0^1 ds \ H(s-f) \int_0^1 ds' \ H(s'-s) \rho(s',t) \]  
(34)

\[ = \int_0^1 ds' \rho(s',t) \int_0^1 ds \ H(s-f) H(s'-s) \]  
(35)

\textbf{Note}\quad H(s-f) H(s'-s) = \begin{cases} 
1 & \text{if } s > f \text{ and } s < s' \\
0 & \text{otherwise}
\end{cases}  
(36)

\begin{align*}
\begin{array}{c}
\vdots
\end{array}
\end{align*}

\begin{align*}
\begin{array}{c}
\text{Only possible for } s' > f, \\
\text{in which case inner integral } = s' - f
\end{array}
\end{align*}

\begin{align*}
\begin{array}{c}
\vdots
\end{array}
\end{align*}

\[ I_2 = \int_f^1 ds' \rho(s',t) [s' - f] \]  
(37)
When $I_1$ is treated similarly,

$$I_1 = \int_0^1 \rho(s', t) [1 - s'] ds' \quad (38)$$

$$+ \int_0^1 ds' \rho(s', t) [s' - f] ds'$$

$$= f (1 - \mu_s(t)) + \int_0^1 ds' \rho(s', t) [s' - f] \quad (39)$$

Adding together $I_1$ and $I_2$,

$$I = (1 - f) \mu_s(t)$$

Returning to eqn. (31)

$$\int_0^1 f(s, t) ds = -V(t)(1 - f) \mu_s(t) \quad (40)$$
The integral of the total flux

\[ \int_{0}^{1} J(s,t) \, ds = \int_{0}^{1} \left[ J_d(s,t) + J_f(s,t) \right] \, ds \]

\[ = \frac{1}{\tau} \left[ 1 - \mu_s(t) \right] - V(t) (1 - f) \mu_s(t) \]

\[ \frac{d}{dt} \mu_s(t) = \frac{1}{\tau} \left[ 1 - \mu_s(t) \right] - V(t) (1 - f) \mu_s(t) \]