Statistical Models of Images, with Application to Denoising and Texture Synthesis

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Statistical Image Models

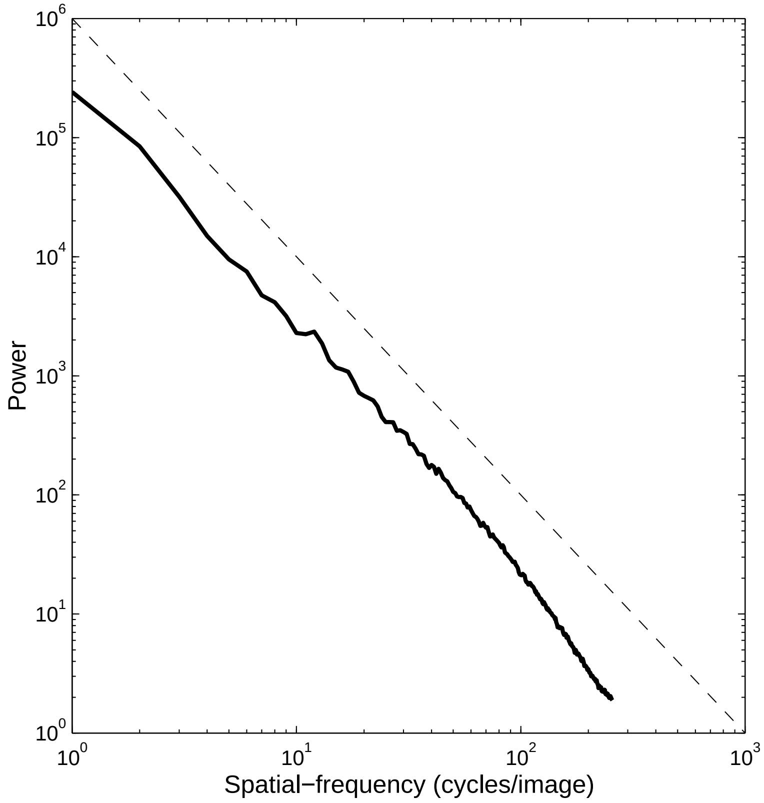
Applications in Image Processing / Graphics:

- Compression
- Restoration
- Enhancement
- Synthesis

Theoretical Neurobiology:

- Ecological optimality principle for early visual processing
- Adaptation / plasticity / learning

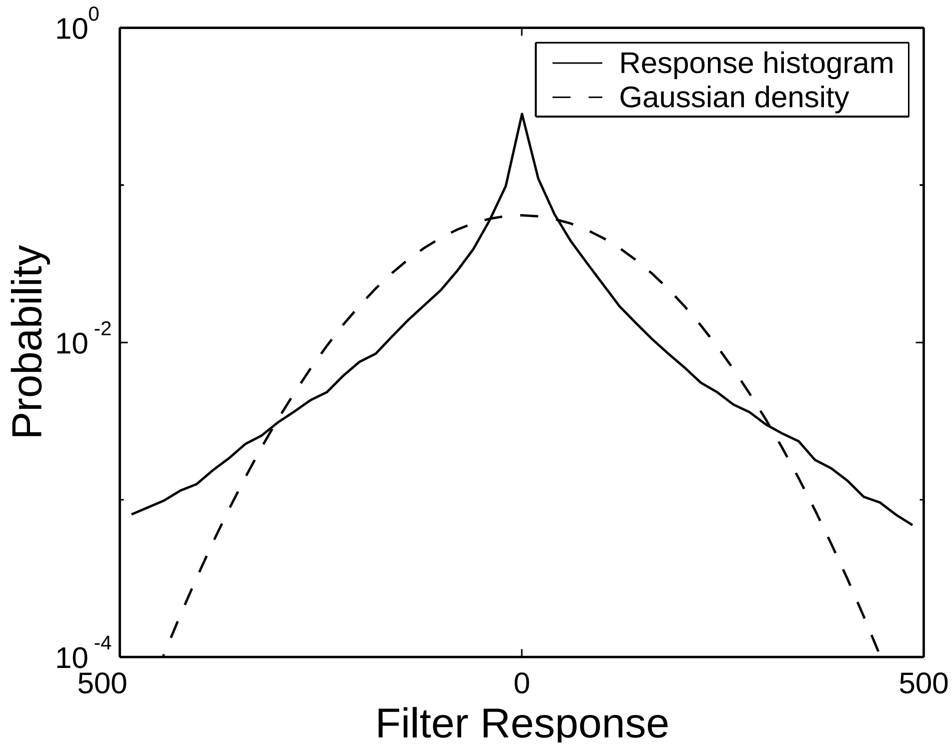
Model I: Gaussian



Power spectra of natural images fall as $1/f^{\alpha}$, $\alpha \sim 2$.

[Field ’87, Ruderman & Bialek ’94, etc]

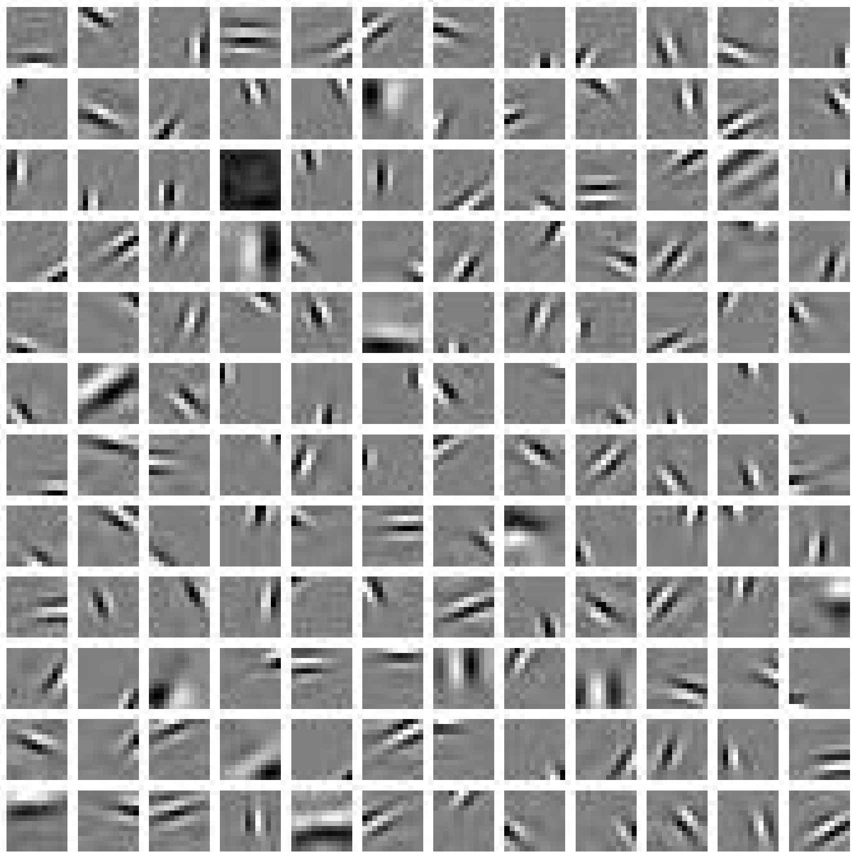
Bandpass Filters Reveal non-Gaussian Behaviors



Marginal densities of bandpass filtered images are non-Gaussian.

[Field87, Mallat89].

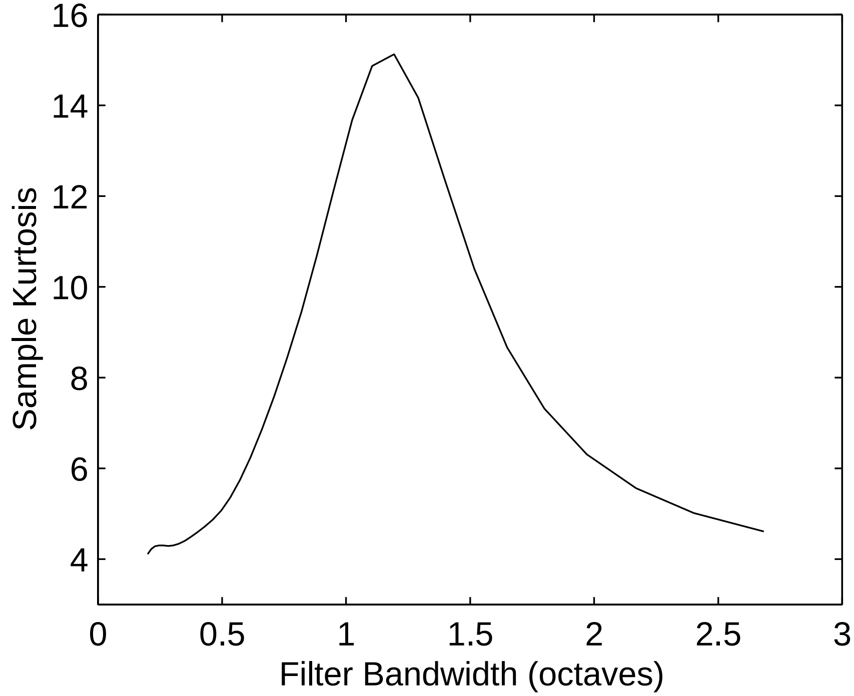
Optimizing non-Gaussianity



Linear operators with maximally independent (or maximally non-Gaussian) responses are oriented bandpass filters

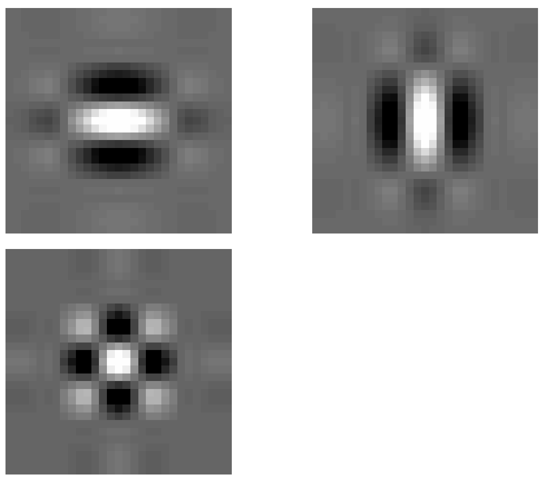
[Bell/Sejnowski ’97; Olshausen/Field ’96]

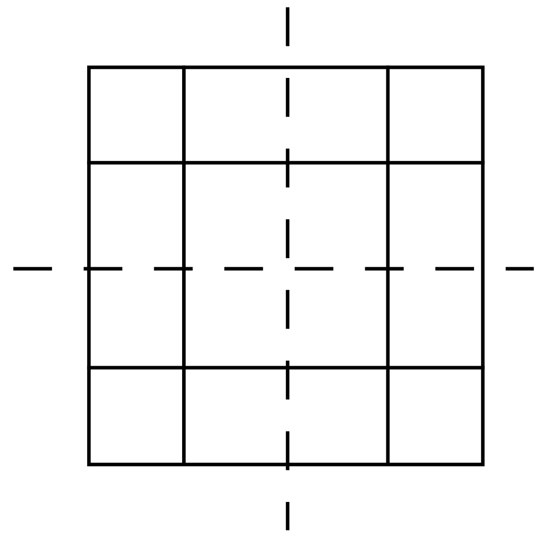
Sample Kurtosis vs. Filter Bandwidth



For most images, maximum is near one octave [after Field, 1987].

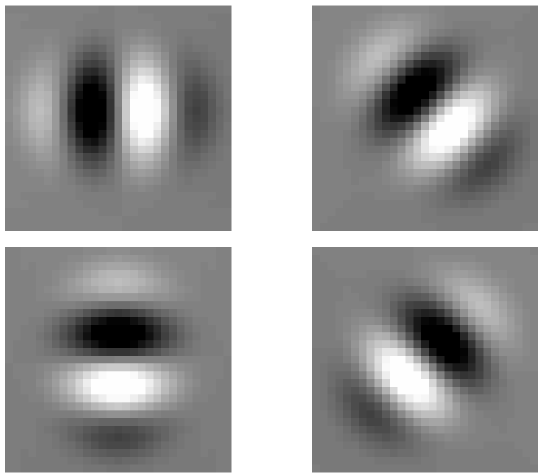
Separable Wavelets

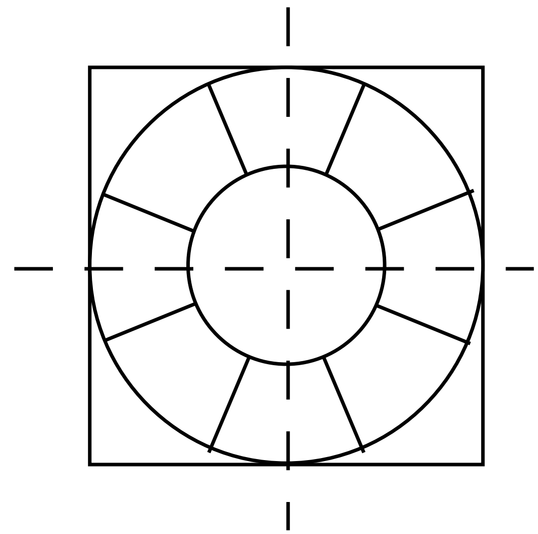




- Basis functions are bandpass filters, related by translation, dilation, modulation.
- Orthogonal.
- Lacking translation- and rotation-invariance.

Steerable Pyramid

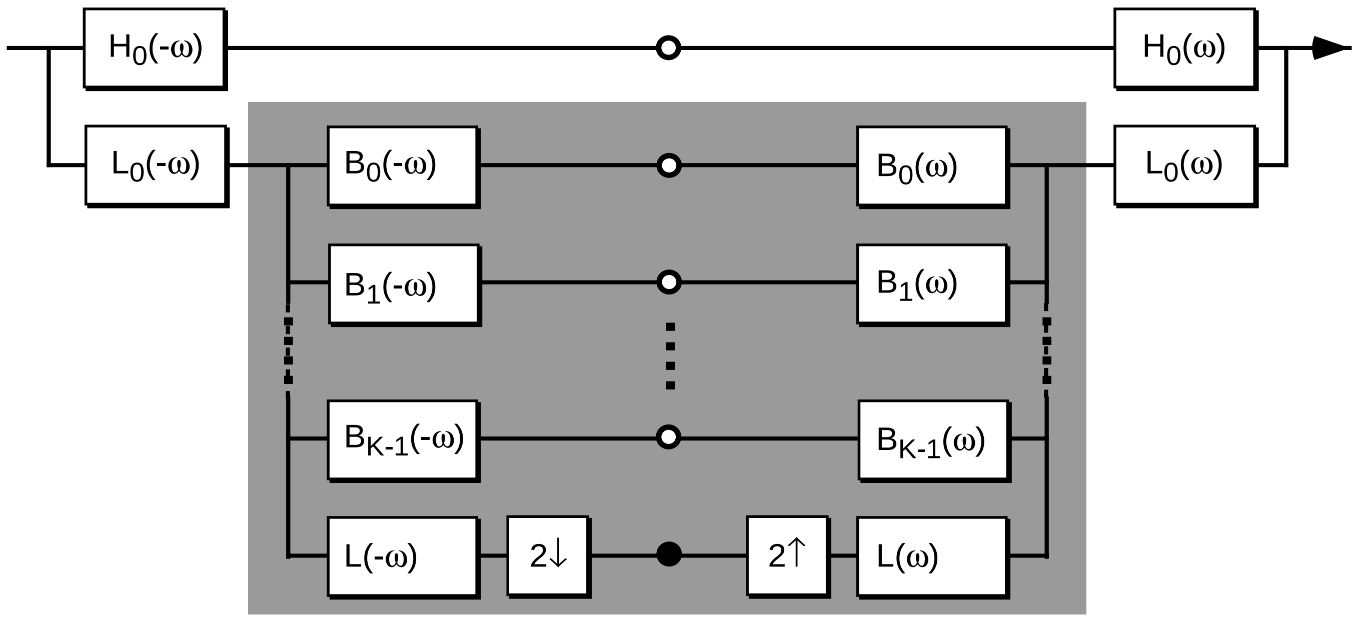




- Basis functions are oriented bandpass filters, related by translation, dilation, **rotation** (directional derivatives, order $K-1$).
- Tight frame, $4K/3$ overcompleteness for K orientations.
- Translation-invariant, rotation-invariant.

[Freeman & Adelson, ’90; Simoncelli et.al., ’91; Freeman & Simoncelli ’95]

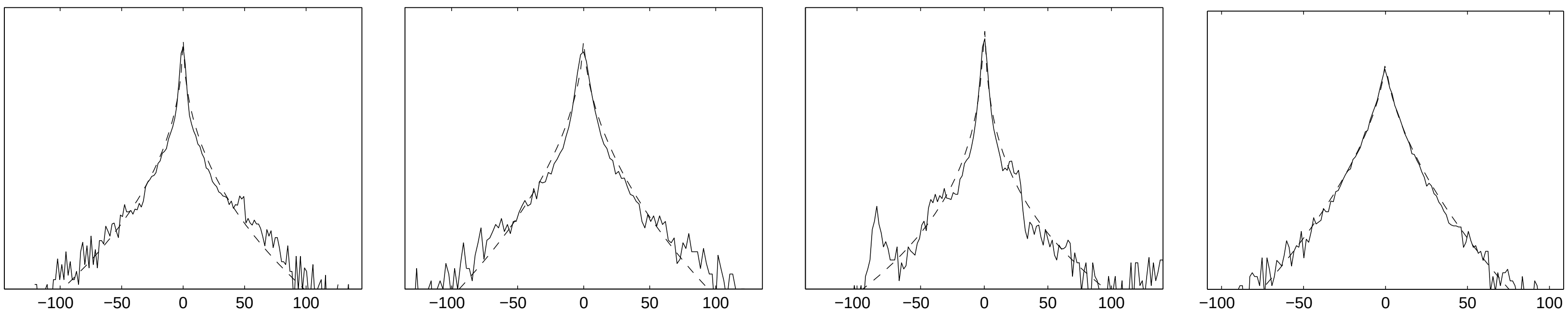
Steerable Pyramid: Block Diagram



Lowpass band is recursively split using central diagram (gray box).

Model II: Wavelet Marginals

Boats | Lena | Toys | Goldhill



Boats	Lena	Toys	Goldhill
$p = 0.62$	$p = 0.56$	$p = 0.52$	$p = 0.60$
$\Delta H = 0.014$	$\Delta H = 0.013$	$\Delta H = 0.021$	$\Delta H = 0.0019$

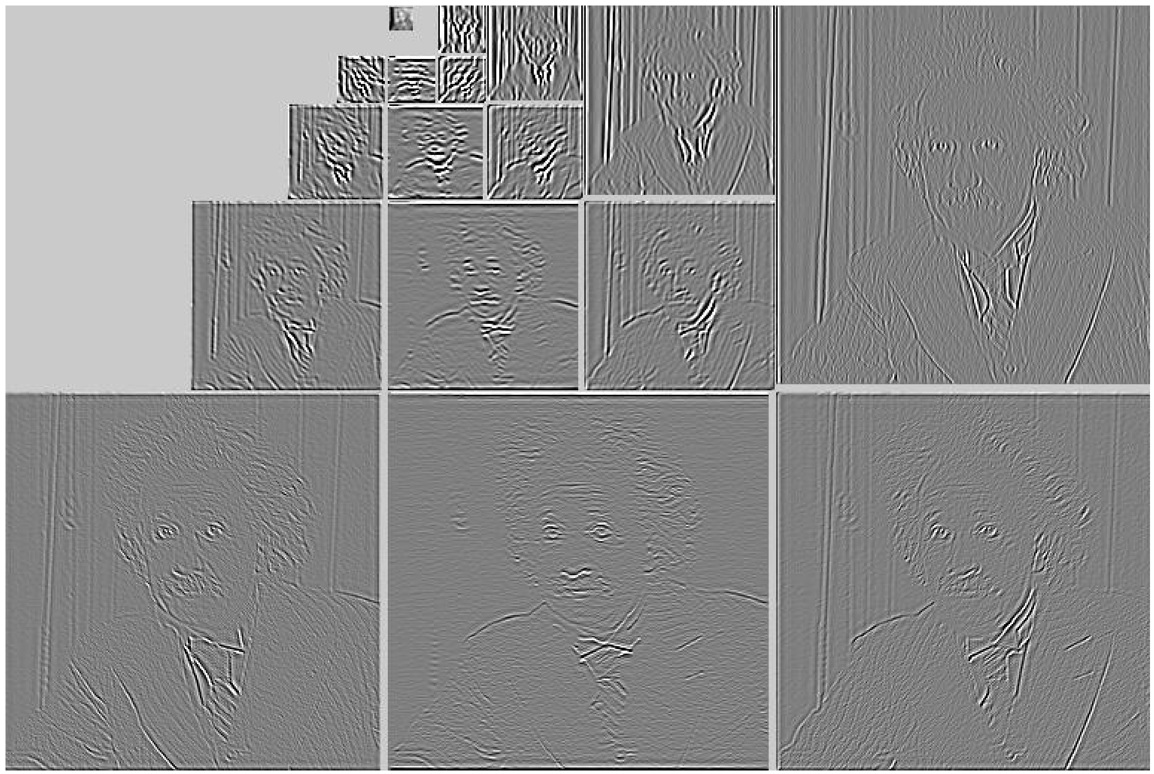
- Coefficient densities well fit by generalized Gaussian

[Mallat ’89; Simoncelli/Adelson ’96]:

$$f(c) \propto e^{-|c/s|^p}, \quad p \in [0.5, 0.8].$$

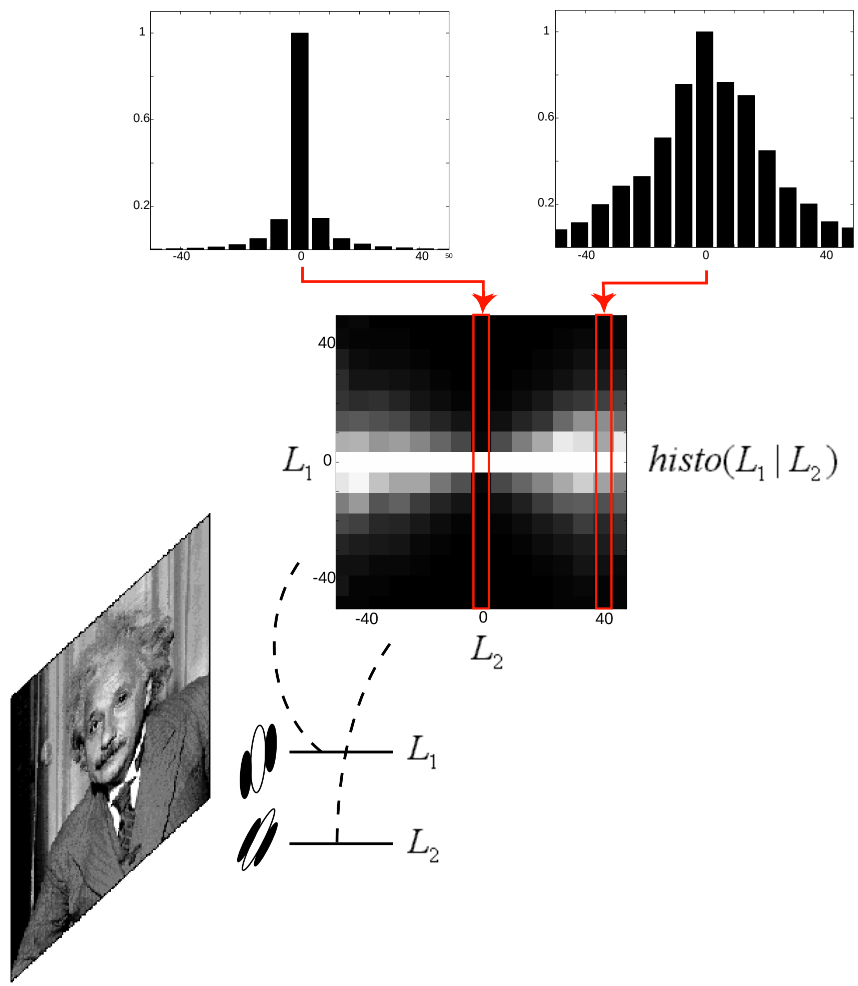
- Non-Gaussianity due to both image content *and* choice of basis.

Coefficient Dependency



Large-magnitude subband coefficients are found at neighboring positions, orientations, and scales.

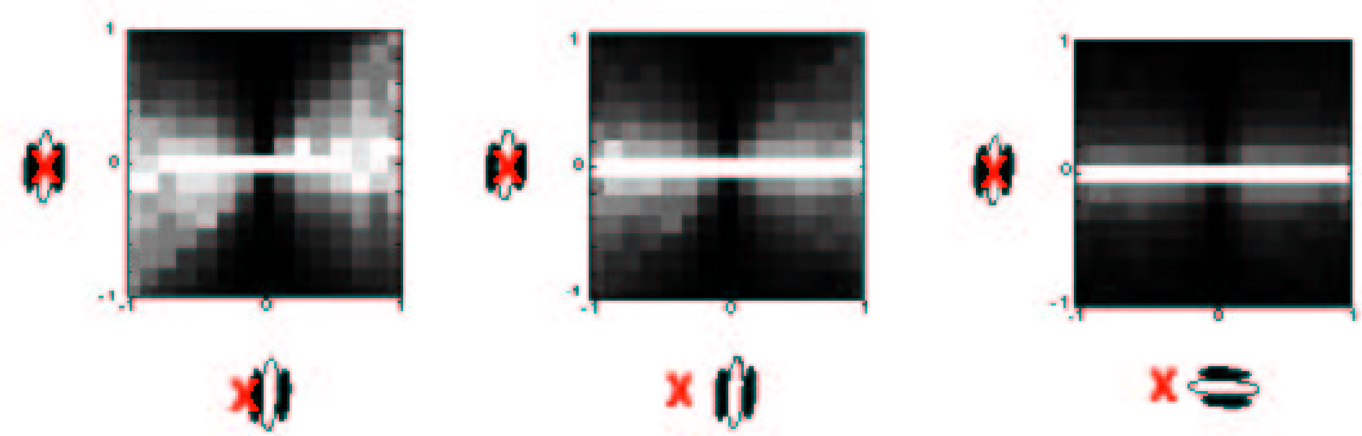
Wavelet Conditional Histogram



- Conditional mean is zero
- But, conditional variance grows with amplitude of L_2

Conditional Histograms

Strength of dependency is different for each pair of filters:



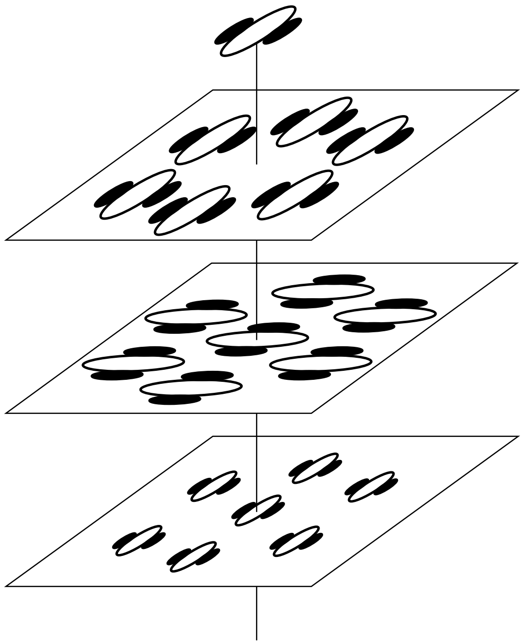
But the form of dependency is highly consistent across a wide range of images.

Model III: Local GSM model

Model generalized neighborhood of coefficients as a Gaussian Scale Mixture (GSM) [Andrews & Mallows ’74]:

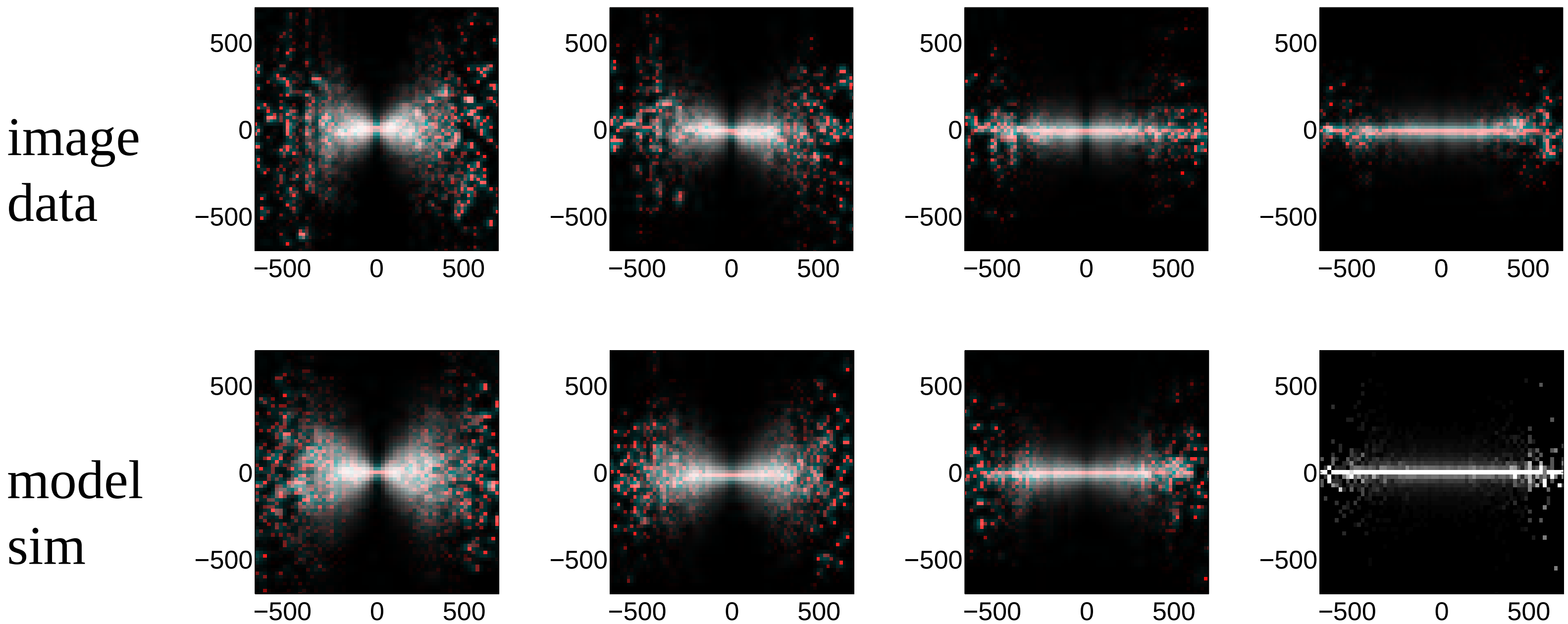
$\vec{x} = \sqrt{z}\,\vec{u}$, where

- z and $\vec{u}$ are independent
- $\vec{x}|z$ is Gaussian, with covariance zC_u
- marginals are always leptokurtotic
- we choose a flat (non-informative) prior on $log(z)$



[Wainwright & Simoncelli, ’99]

GSM Simulations



Conditional Histograms of pairs of coefficients with different spatial separations.

Denoising I (Gaussian model)

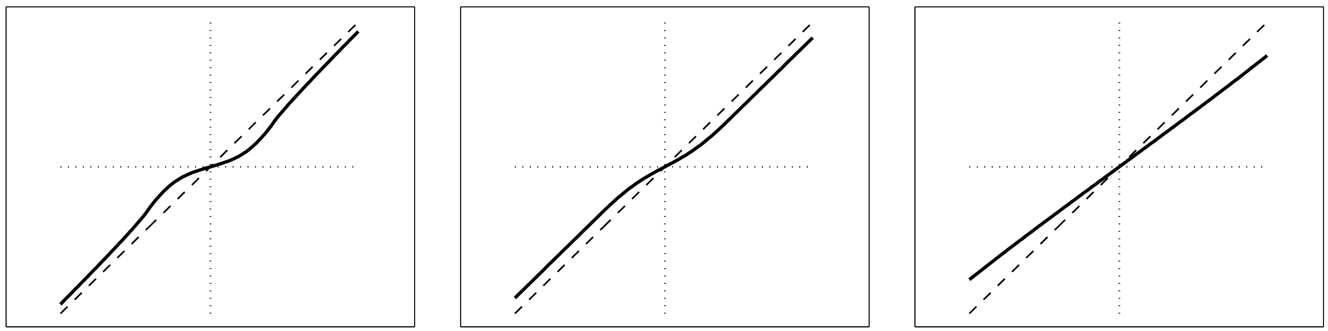
$$y = x + w, \qquad \text{where } w \text{ is Gaussian, white.}$$

y is an observed transform coefficient.

Bayes least squares solution:

$$\mathbb{E}(x|y) = \frac{\sigma_x^2}{\sigma_x^2 + \sigma_w^2} y$$

Denoising II (marginal model)



$p = 0.5$ $\qquad$ $p = 1.0$ $\qquad$ $p = 2.0$

Bayes least squares solution:

$$\mathbb{E}(x|y) = \frac{\int dx\ \mathcal{P}(y|x)\ \mathcal{P}(x)\ x}{\int dx\ \mathcal{P}(y|x)\ \mathcal{P}(x)}$$

No closed-form expression with generalized Gaussian prior, but numerical computation is reasonably efficient.

[Simoncelli & Adelson, '96]

Denoising III (GSM model)

$$\mathbb{E}(x|\vec{y}) = \int dz\, \mathcal{P}(z|\vec{y})\, \mathbb{E}(x|\vec{y},z)$$
$$= \int dz\, \mathcal{P}(z|\vec{y}) \quad \left[zC_u(zC_u + C_w)^{-1}\vec{y}\right]_{\text{ctr}}$$

where

$$\mathcal{P}(z|\vec{y}) = \frac{\mathcal{P}(\vec{y}|z)\ \mathcal{P}(z)}{\int dz\ \mathcal{P}(\vec{y}|z)\ \mathcal{P}(z)}, \qquad \mathcal{P}(\vec{y}|z) = \frac{\exp(-\vec{y}^T(zC_u + C_w)^{-1}\vec{y}/2)}{\sqrt{(2\pi)^N|zC_u + C_w|}}$$

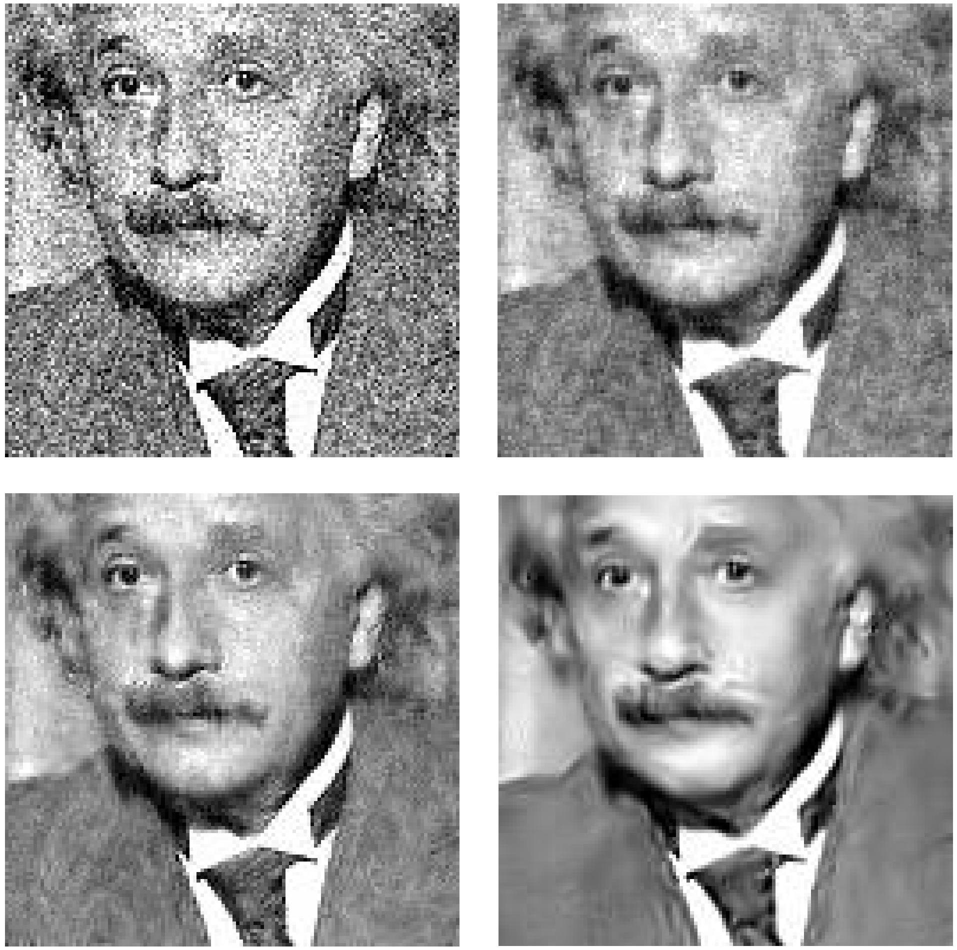
Numerical computation of solution is reasonably efficient if one jointly diagonalizes C_u and C_w ...

[Portilla et.al., '01]

Denoising Simulation: Face

noisy
(4.8)

I-linear
(10.61)

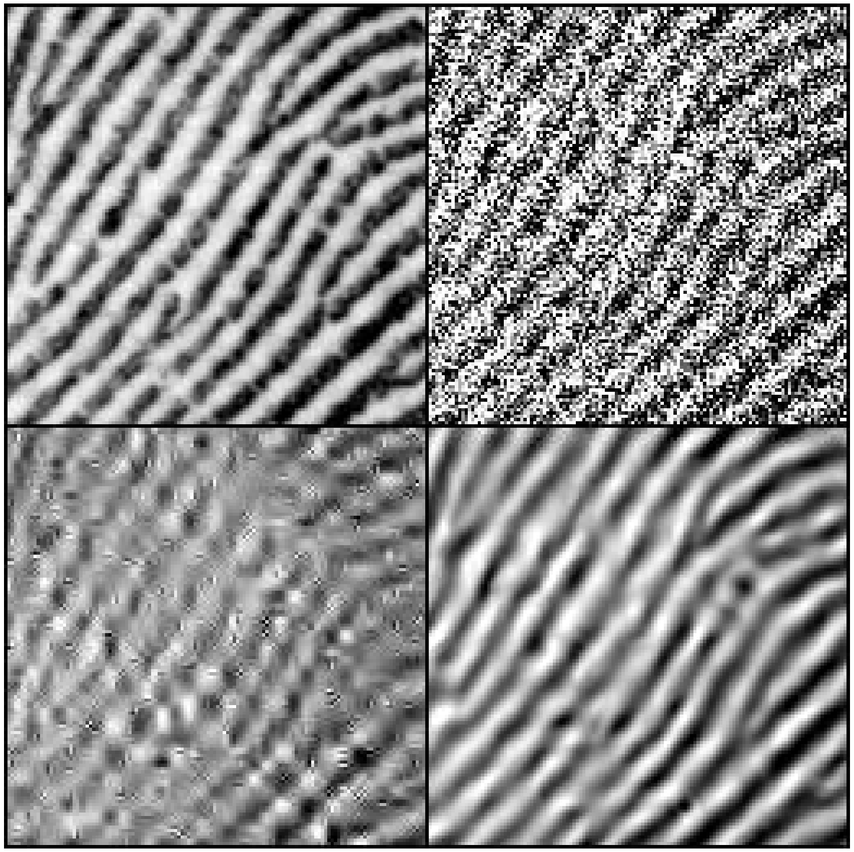


II-marginal
(11.98)

III-GSM
nbd: $5 \times 5 + p$
(13.60)

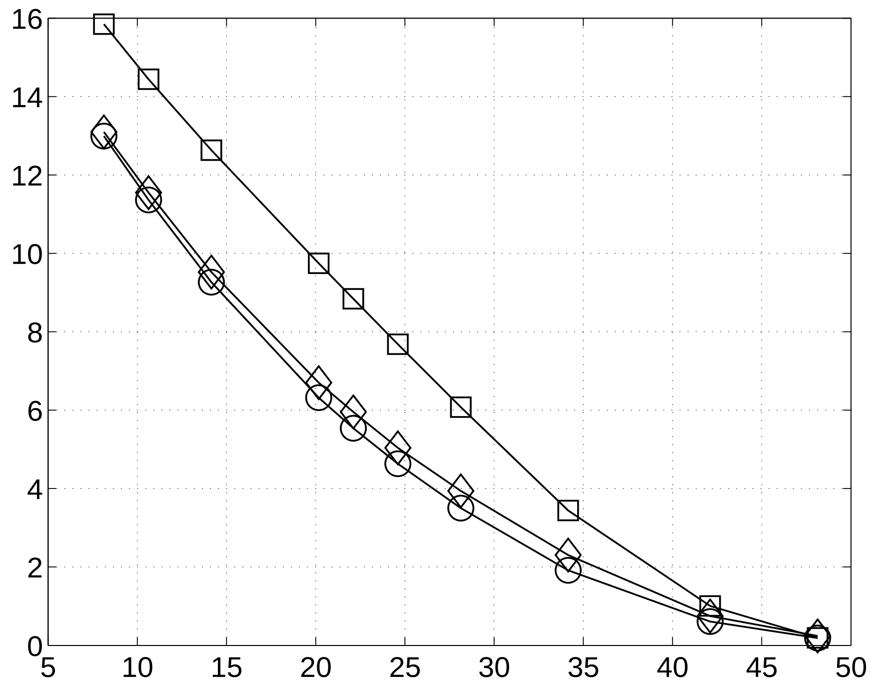
- Semi-blind (all parameters estimated except for σ_w).
- All methods use same steerable pyramid decomposition.
- SNR (in dB) shown in parentheses.

Denoising Simulation: Fingerprint



- PSNR shown in parentheses.
- Both methods use same steerable pyramid decomposition.
- Joint statistics capture oriented structures.

Denoising Comparison

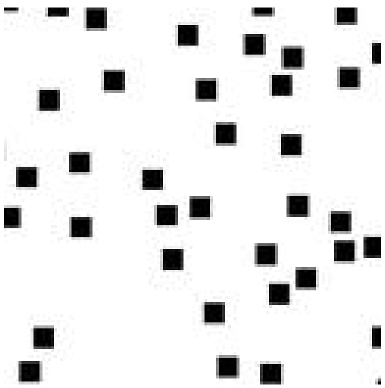


PSNR improvement as a function of noise level, averaged over three images:

- squares: GSM
- triangles: MatLab *wiener2*, optimized neighborhood [Lee, ’80]
- circles: soft thresholding, optimized threshold [Donoho, ’95]

Example Texture Types

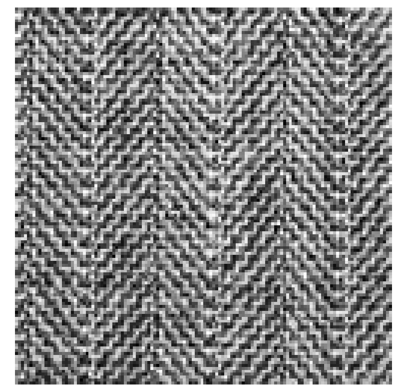
structured



random

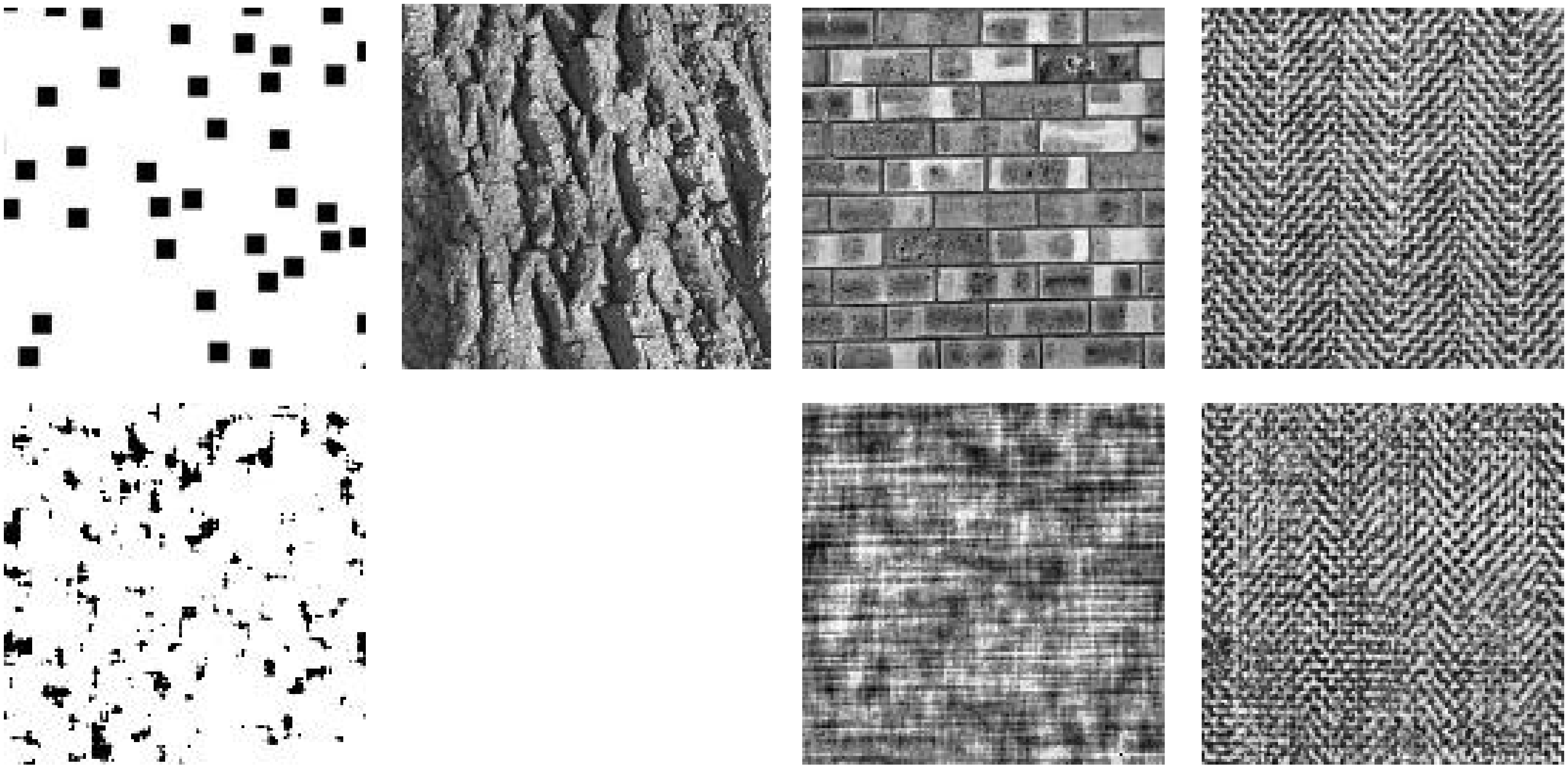
periodic

2nd-order



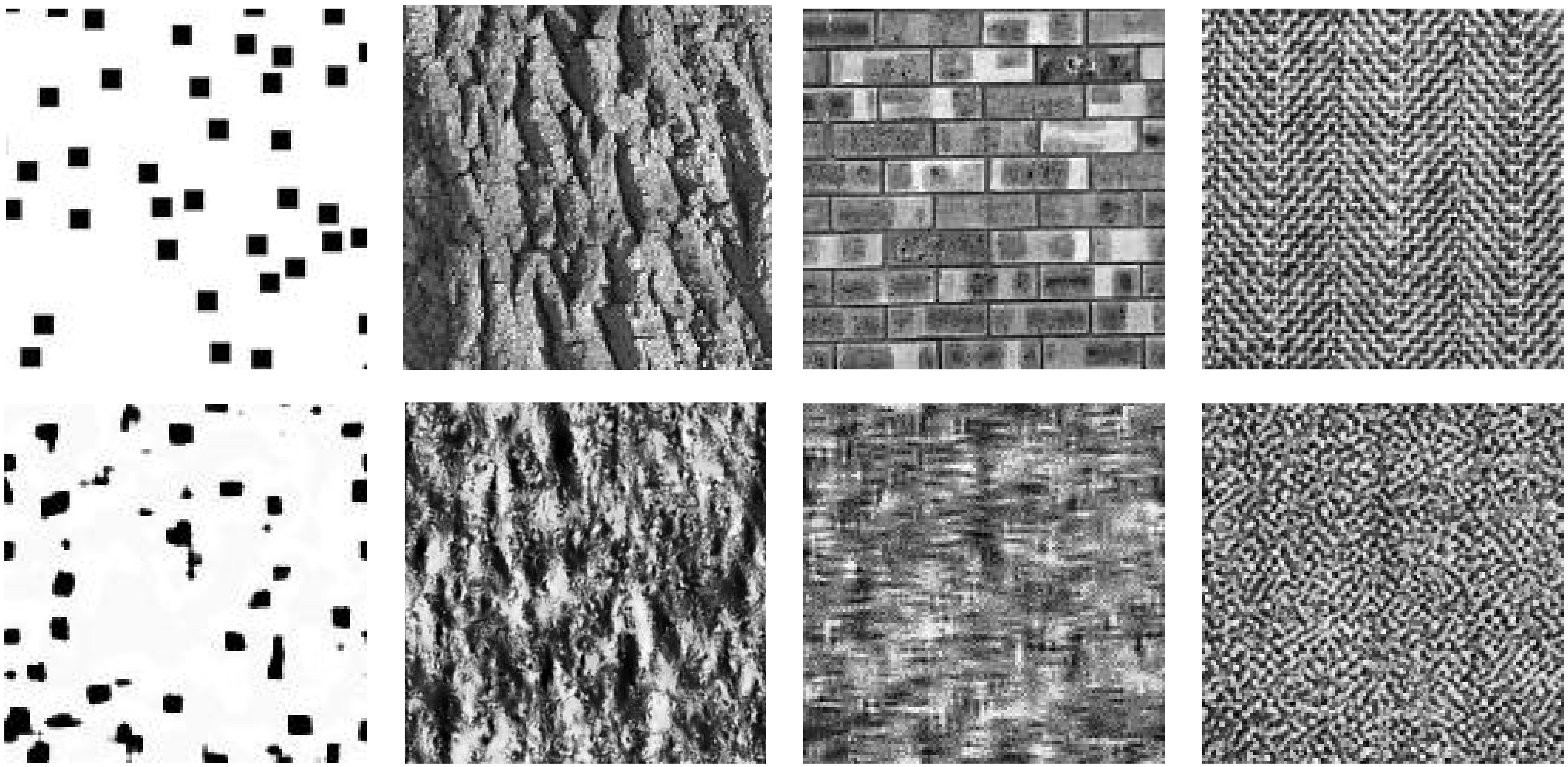
Can we derive a statistical model (and sampling technique) to represent all of these?

Synthesis: Gaussian model



Captures periodicity.

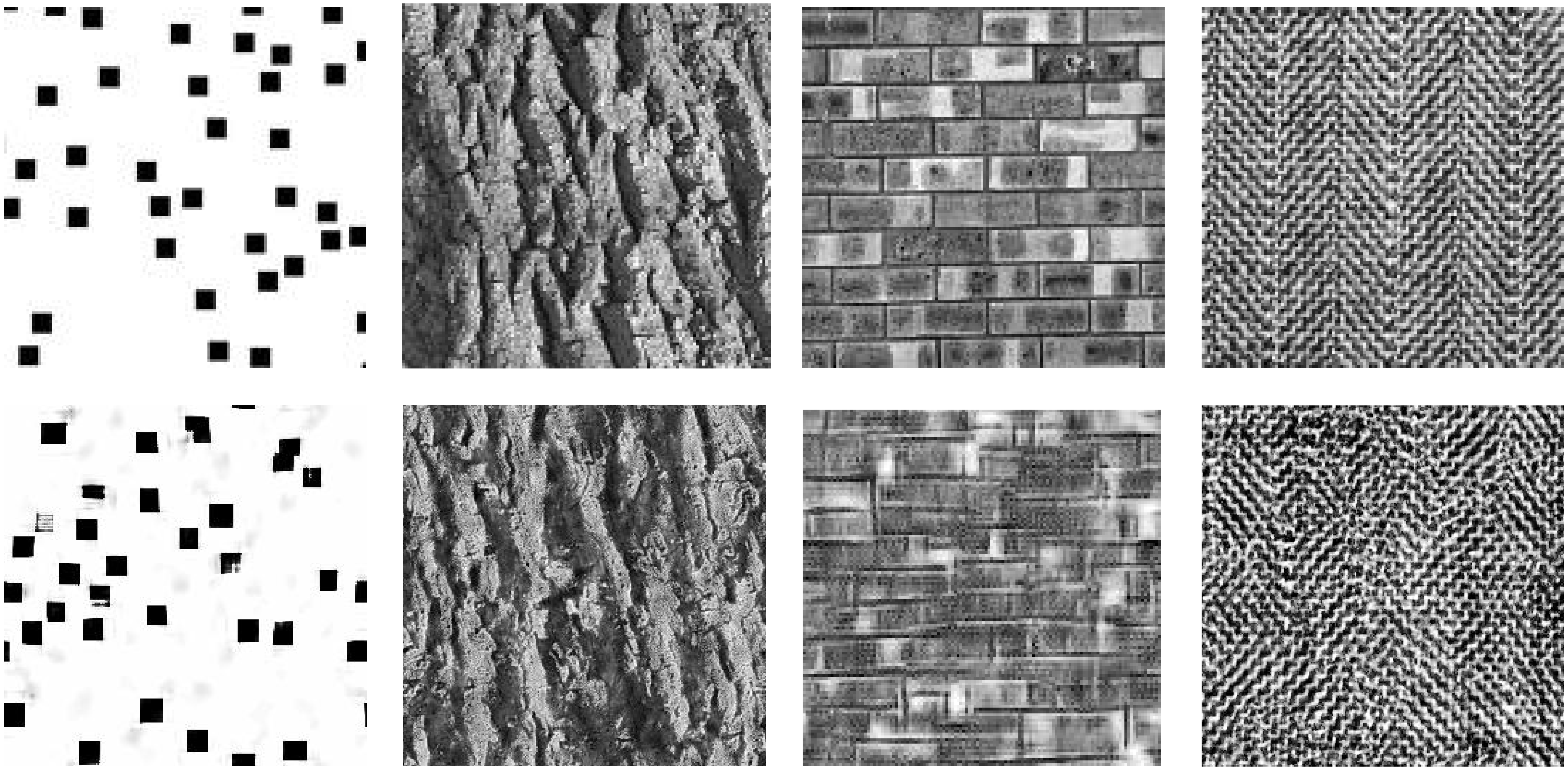
Synthesis: Wavelet marginal model



Captures some local structure.

[Heeger & Bergen, '95]

Synthesis: GSM model



[Portilla & Simoncelli, '00]

Credits

Local Gaussian Scale mixtures: Martin Wainwright (MIT)

Global Tree Model: Martin Wainwright & Allan Willsky (MIT)

Denoising: Javier Portilla (U. Granada), Vasily Strela (Drexel U.), Martin Wainwright (MIT)

Texture Analysis/Synthesis: Javier Portilla (U. Granada)

Compression: Robert Buccigrossi (U Pennsylvania)

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