Notes on Steerable Filters
Psych 267/CS 348D/EE 365
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This handout reviews the work of many authors in what has commonly come to be known as steerable filters (a phrase coined by Freeman and Adelson in [3]). Related earlier work can be also be found in [2, 5, 6], and some more recent work can also be found in, among others, [1, 4, 7, 8, 9, 10, 11].

1   Steerability of First-Order Directional Derivatives

The simplest illustration of a steerable function is the first-order directional derivative of a two-dimensional Gaussian. Although we will only consider Gaussians, the principle of steerability may be extended to any differentiable function. For notational simplicity we will consider a unit-variant Gaussian and ignore the $\frac{1}{\sqrt{2\pi}}$ scaling constant:

$$g(x, y) = e^{-(x^2+y^2)/2}. \tag{1}$$

Lets begin by considering the first-order horizontal derivative (i.e., in x) of $g(x, y)$, given by:

$$
\begin{aligned}
g_x(x, y) &= \frac{\partial g(x, y)}{\partial x} \\
&= -x e^{-(x^2+y^2)/2}.
\end{aligned}
\tag{2}
$$

Illustrated in Figure 1 are sampled versions of these two functions, Equations (1) and (2), respectively.

Our goal is to now show that the directional derivative is steerable: that is, it can be synthesized at *any* orientation from a linear combination of the same function at a fixed set of orientations. In that we are interested in rotations of this function, it is perhaps more natural to consider these functions in polar coordinates r and θ where: $x = r\cos(\theta)$, $y = r\sin(\theta)$, and so $r^2 = x^2 + y^2$. In polar coordinates, the horizontal directional derivative (Equation (2)) is given by:

$$g_x(r, \theta) = -r e^{-r^2/2} \cos(\theta). \tag{3}$$

Note that this function is polar-separable, that is, it is a product of a radial ($-re^{-r^2/2}$) component and angular ($\cos(\theta)$) component. Since we are interested in rotations of this function, lets first consider a copy of the function $g_x(\cdot)$ rotated by $\pi/2$ (see Figure 1). Substituting $\theta - \pi/2$ into Equation (3) and using a basic trigonometric identity ($\cos(A - B) =$

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[1]Modified from a handout written by H. Farid, University of Pennsylvania

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$\cos(A)\cos(B) + \sin(A)\sin(B))$ gives:

$$
\begin{aligned}
g_x(r, \theta - \pi/2) &= -re^{-r^2/2}\cos(\theta - \pi/2) \\
&= -re^{-r^2/2}(\cos(\theta)\cos(\pi/2) + \sin(\theta)\sin(\pi/2)) \\
&= -re^{-r^2/2}\sin(\theta). \tag{4}
\end{aligned}
$$

Note that this function differs from the horizontal derivative in the angular portion only, the radial portions are identical. We should of course not be surprised that the horizontal derivative rotated by $\pi/2$ is simply the vertical derivative (i.e., in y) of $g(\cdot)$:

$$
\begin{aligned}
g_y(x, y) &= \frac{\partial g(x, y)}{\partial y} \\
&= -ye^{-(x^2+y^2)/2}, \text{ and in polar coordinates,} \\
g_y(r, \theta) &= -re^{-r^2/2}\sin(\theta), \tag{5}
\end{aligned}
$$

which of course is in agreement with Equation (4). The significance of this is that the directional derivative at any orientation is simply a rotated copy of the same function. This is one of two conditions that must hold in order for a function to be steerable, the second condition is derived next.

Consider now the directional derivative $g_x(\cdot)$ rotated to an *arbitrary* angle, α (see Figure 1). As before, substitute $\theta - \alpha$ into Equation (3) and use the same trigonometric identity:

$$
\begin{aligned}
g_\alpha(r, \theta) &= -re^{-r^2/2}\cos(\theta - \alpha) \\
&= -re^{-r^2/2}(\cos(\theta)\cos(\alpha) + \sin(\theta)\sin(\alpha)) \\
&= \cos(\alpha)(-re^{-r^2/2}\cos(\theta)) + \sin(\alpha)(-re^{-r^2/2}\sin(\theta)) \\
&= \cos(\alpha)g_x + \sin(\alpha)g_y. \\
&= (\,\cos(\alpha) \quad \sin(\alpha)\,)\begin{pmatrix} g_x \\ g_y \end{pmatrix} \\
&= \mathbf{v}(\alpha) \cdot \mathbf{b}, \tag{6}
\end{aligned}
$$

Equation 6 embodies the principle of steerability. In particular, the first-order directional derivative of $g(\cdot)$ can be synthesized at an *arbitrary* orientation from a linear combination of the function oriented at 0 and $\pi/2$ ($g_x(\cdot)$ and $g_y(\cdot)$, respectively). The functions $g_x(\cdot)$ and $g_y(\cdot)$ are referred to as the *basis set* and $\cos(\alpha)$ and $\sin(\alpha)$ are referred to as the *interpolation functions*. The vector $\mathbf{v}(\alpha) = (\cos(\alpha), \sin(\alpha))$ contains the values of the interpolation functions for a particular angle α. Abusing notation, we have written the basis set as a "vector" $\mathbf{b}$; each element of this "vector" is one of the directional derivatives (g_x and g_y), that is, each is a function of x and y.

Behind the mathematical formulation of steerability there is some simple intuition. Recall that in polar coordinates the directional derivative at arbitrary orientations differed only in their angular portion, and in the case of the Gaussian, this angular component
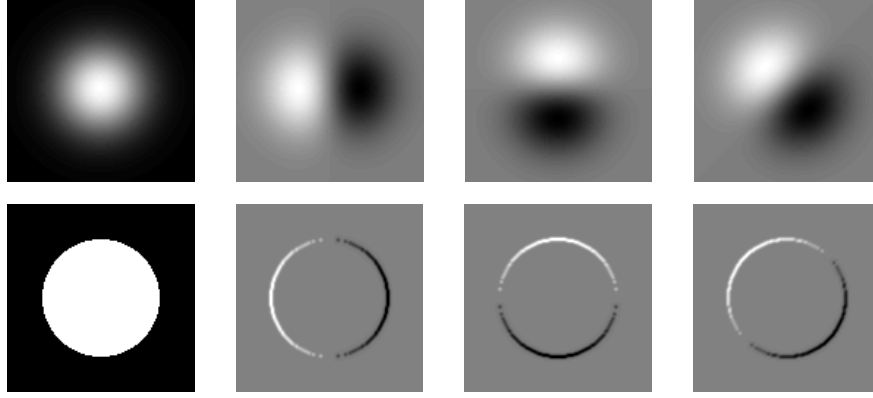
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Figure 1: Steerability of the first-order directional derivatives and their responses. Illustrated along the **top row**, from left to right is: (1) a sampled 2d Gaussian, $g(x,y)$; (2) the horizontal directional derivative, $g_x(x,y)$ (i.e., oriented at 0); (3) the vertical derivative, $g_y(x,y)$ (i.e., oriented at $\pi/2$); and (4) the directional derivative "steered" to $\pi/4$: this directional derivative was synthesized from a linear combination of the horizontal and vertical directional derivatives. Illustrated along the **bottom row**, from left to right is: (1) a disc image; (2) the result of applying the horizontal directional derivative; (3) the result of applying the vertical directional derivative; and (4) the "steered" response of a directional derivative oriented at $\pi/4$: this image was computed from a linear combination of the horizontal and vertical derivative images, the actual filter (shown above) was never actually synthesized or applied!

was a single frequency sinusoid (Equations (3) and (4)). Thus, phase shifts in the angular component amounts to rotations of the directional derivative (e.g., the horizontal derivative's angular component is $\cos(\theta)$, and the vertical derivative's angular portion is $\sin(\theta) = \cos(\theta - \pi/2)$ - a phase shift of $\pi/2$). Now, a single frequency sinusoid has two degrees of freedom, the amplitude and phase, so it should not be surprising that it can be fully characterized by a basis set of size two. We can also draw upon the Nyquist sampling rate for further intuition. In particular, according to Nyquist, a function bandlimited to consist of n frequency harmonics, can be sampled at a rate of $2n$ without any loss of information. With respect to our directional derivatives, the angular portion consists of a single frequency sinusoid, thus we require only two samples to fully characterize the function. The further interested reader is referred to [4] for a presentation based on Lie groups and [1] for a presentation based on cartesian tensor calculus.

The steerability of the directional derivative is not dependent on the selection of the basis set being oriented at 0 and $\pi/2$, the basis set can be chosen at any two distinct orientations resulting only in different interpolation functions. In particular, following the same formulation as above, a basis set oriented at θ_1 and θ_2 is given by:

$$
\begin{aligned}
g_{\theta_1}(r,\theta) &= -re^{-r^2/2}(\cos(\theta)\cos(\theta_1) + \sin(\theta)\sin(\theta_1)) &\quad (7)\\
g_{\theta_2}(r,\theta) &= -re^{-r^2/2}(\cos(\theta)\cos(\theta_2) + \sin(\theta)\sin(\theta_2)) &\quad (8)
\end{aligned}
$$

These equations can be writen in matrix notation as follows:

$$\mathbf{b'} = \mathbf{M}\,\mathbf{b}, \tag{9}$$

where

$$\mathbf{b'} = \begin{pmatrix} g_{\theta_1} \\ g_{\theta_2} \end{pmatrix} \quad \mathbf{M} = \begin{pmatrix} \cos(\theta_1) & \sin(\theta_1) \\ \cos(\theta_2) & \sin(\theta_2) \end{pmatrix} \quad \mathbf{b} = -re^{-r^2/2}\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \tag{10}$$

The angular component of the directional derivative oriented at an arbitrary angle α can be written in terms of the same basis set $\mathbf{b}$:

$$g_\alpha = \mathbf{v}(\alpha)\cdot\mathbf{b} \quad \text{where} \quad \mathbf{v}(\alpha) = \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \end{pmatrix} \tag{11}$$

Finally, this function can be expressed with respect to the basis oriented at θ_1 and θ_2 by combining the above two equations:

$$\begin{aligned} g_\alpha &= \left[\mathbf{v}^t(\alpha)\,\mathbf{M}^{-1}\right]\begin{pmatrix} g_{\theta_1} \\ g_{\theta_2} \end{pmatrix} \\ &= \mathbf{v'}(\alpha)\cdot\mathbf{b'}, \end{aligned} \tag{12}$$

where the first term in square brackets evaluates to a 2-vector $\mathbf{v'}$ containing the new interpolation functions, and the "vector" $\mathbf{b'} = (g_{\theta_1}, g_{\theta_2})^t$ contains the new basis set. Note that $\mathbf{M}$ is invertible if and only if $\theta_1 \neq tg_2 + k\pi$ for integer k; in other words, any two directional derivatives will do unless they are the same or unless they are negatives/reflections of one another. Note also that for the canonical basis oriented at 0 and $\pi/2$, the matrix $\mathbf{M}$ in the above equation reduces to the identity matrix, leaving an expression identical to our previous formulation (Equation (6)).

To review, the first-order directional derivative of a 2d Gaussian is steerable with a basis set of size two. This result was based on two key observations: (1) the directional derivative at arbitrary orientations are simply rotated copies of the same function; and (2) when considered in polar coordinates, the angular component of the directional derivative consists of a single frequency, two degree of freedom, sinusoid. As a result, the first-order directional derivative at any orientation can be synthesized from a linear combination of the same function at any two distinct orientations.

Within the image processing community, the directional derivative is a commonly used filter (typically applied via a convolution to a digital image). To this end, the steerability of the directional derivative is convenient for the *synthesis* of filters. We can however take the principle of steerability one step further and show that the filter *responses* are also steerable - a not all together surprising fact given that convolution is a linear operation. In particular, consider the application of the horizontal and vertical directional derivative to an image, $a(x, y)$.

$$a_x(x, y) = g_x(x, y) \star a(x, y) \qquad \text{and} \qquad a_y(x, y) = g_y(x, y) \star a(x, y), \tag{13}$$

where $\star$ denotes the convolution operator ($f(x, y) = g(x, y) \star h(x, y) = \sum_{u,v=-\infty}^{\infty} g(u, v)h(x-u, y-v)$). In the same way in which the *filter* $g_\alpha(\cdot)$ can be synthesized from a linear combination of the basis set $g_x(\cdot)$ and $g_y(\cdot)$, the *response* to a directional derivative oriented at

α can also be synthesized from a linear combination of the pair of derivative images $a_x(\cdot)$ and $a_y(\cdot)$ (Figure 1):

$$
\begin{aligned}
a_\alpha(x, y) &= g_\alpha(x, y) \star a(x, y) \\
&= [\cos(\alpha)g_x(x, y) + \sin(\alpha)g_y(x, y)] \star a(x, y) \\
&= [\cos(\alpha)g_x(x, y) \star a(x, y)] + [\sin(\alpha)g_y(x, y)) \star a(x, y)] \\
&= \cos(\alpha)a_x(x, y) + \sin(\alpha)a_y(x, y).
\end{aligned} \tag{14}
$$

Note that the interpolation functions ($\cos(\alpha)$ and $\sin(\alpha)$) for synthesizing the filters and the filter responses are identical, and the basis set ($a_x(\cdot)$ and $a_y(\cdot)$) is now the *images* generated by convolving with the basis filter set.

2 Steerability of Higher-Order Directional Derivatives

The previous showed that the first-order directional derivative of a Gaussian is steerable with a basis set of size two (in fact this result extends to any differentiable function). Here we will show that higher-order directional derivatives are also steerable: an nth-order directional derivative is steerable with a basis set of size $n + 1$. For this reason, the steerability of higher-order directional derivatives is presented in the Fourier domain, and it is expressed in terms of the higher-order separable derivatives.

We adopt the following notation:

$g(x, y)$	prefilter (e.g., 2d Gaussian)
$g_x^{(n)} = \frac{\partial^n}{\partial x^n}g$	n^{th} derivative of g in the x-direction
$g_y^{(n)} = \frac{\partial^n}{\partial y^n}g$	n^{th} derivative of g in the y-direction
$g_u^{(n)}$	n^{th} derivative of g in the direction of the unit vector $\hat{\mathbf{u}}$
$\hat{\mathbf{u}} = (u_x, u_y)^t$	unit vector
$\mathbf{w} = (w_x, w_y)^t$	spatial frequency
$\hat{\mathbf{w}} = (\hat{w}_x, \hat{w}_y)^t$	unit vector parallel to $\mathbf{w}$
$\mathcal{F}\{g(x, y)\} = G(\mathbf{w})$	Fourier transform of g
$\mathcal{F}\{\frac{\partial^n}{\partial x^n}g\} = G_x^{(n)}(\mathbf{w}) = (-jw_x)^n G(\mathbf{w})$	Fourier transform of $g_x^{(n)}$

The last line follows from the derivative property of the Fourier transform: the Fourier transform of the derivative of a function equals a complex ramp $-jw$ times Fourier transform of the original function. For an intuition for the derivative property, recall that the derivative of $\frac{\partial}{\partial x}\cos(wx) = -w\sin(wx)$.

We begin by reformulating the steerability of the first-order directional derivative and then show how this new formulation extends naturally to second- and higher-order derivatives. Define $\hat{\mathbf{w}}$ to be a unit vector pointing in the direction of the spatial frequency $\mathbf{w}$ and $\hat{\mathbf{u}}_x$ is a unit vector pointing along the x-frequency axis, so that:

$$
\mathbf{w} = |\mathbf{w}|\hat{\mathbf{w}} \tag{15}
$$

$$
w_x = \mathbf{w} \cdot \hat{\mathbf{u}}_x = |\mathbf{w}|\hat{\mathbf{w}} \cdot \hat{\mathbf{u}}_x. \tag{16}
$$

Then the first derivative in the x-direction, written in the Fourier domain, is:

$$
\begin{aligned}
G_x^{(1)}(\mathbf{w}) &= -jw_x G(\mathbf{w}) \\
&= -j\left(\hat{\mathbf{w}} \cdot \hat{\mathbf{u}}_x\right) |\mathbf{w}|\, G(\mathbf{w}),
\end{aligned}
\tag{17}
$$

Likewise, the first derivative in an arbitrary direction (specified by the unit vector $\hat{(\mathbf{u})}$) is:

$$
\begin{aligned}
G_u^{(1)}(\mathbf{w}) &= -j\left(\hat{\mathbf{w}} \cdot \hat{\mathbf{u}}\right) |\mathbf{w}|\, G(\mathbf{w}) \\
&= -ju_x w_x G(\mathbf{w}) - ju_y w_y G(\mathbf{w}) \\
&= \begin{pmatrix} u_x & u_y \end{pmatrix} \begin{pmatrix} G_x^{(1)} \\ G_y^{(1)} \end{pmatrix}
\end{aligned}
\tag{18}
$$

Back in the space domain:

$$
g_u^{(1)}(x, y) = \begin{pmatrix} u_x & u_y \end{pmatrix} \begin{pmatrix} g_x^{(1)} \\ g_y^{(1)} \end{pmatrix},
\tag{19}
$$

which is the steering equation (Equation 6). The first vector contains the interpolation functions $(u_x, u_y) = (\cos(\alpha), \sin(\alpha))$ and the second "vector" contains the basis functions (x- and y- first derivatives of g).

We can also express the second derivative in an arbitrary direction as a sum of three separable derivatives. In the frequency domain:

$$
\begin{aligned}
G_u^{(2)}(\mathbf{w}) &= \left[-j\left(\hat{\mathbf{w}} \cdot \hat{\mathbf{u}}\right) |\mathbf{w}|\right]^2 G(\mathbf{w}) \\
&= -(\hat{\mathbf{w}} \cdot \hat{\mathbf{u}})^2 |\mathbf{w}|^2\, G(\mathbf{w}) \\
&= -(\hat{w}_x \hat{u}_x + \hat{w}_y \hat{u}_y)^2 |\mathbf{w}|^2\, G(\mathbf{w}) \\
&= -(\hat{w}_x^2 \hat{u}_x^2 + \hat{w}_y^2 \hat{u}_y^2 + 2\hat{w}_x \hat{w}_y \hat{u}_x \hat{u}_u) |\mathbf{w}|^2\, G(\mathbf{w}) \\
&= -\left[\sum_{k=0}^{2} \frac{2!}{k!\,(2-k)!}\, \hat{u}_x^k\, \hat{u}_y^{2-k}\, \hat{w}_x^k\, \hat{w}_y^{2-k}\right] |\mathbf{w}|^2\, G(\mathbf{w})
\end{aligned}
\tag{20}
$$

Back in the space domain:

$$
g_u^{(2)}(x, y) = \sum_{k=0}^{2} \left[\frac{2!}{k!\,(2-k)!}\, \hat{u}_x^k\, \hat{u}_y^{2-k}\, \frac{\partial^2\, g(x, y)}{\partial x^k\, \partial y^{2-k}}\right]
\tag{21}
$$

The nth-order directional derivative in the direction of $\hat{\mathbf{u}}$ can likewise be expressed as a sum of $n+1$ separable derivatives:

$$
\begin{aligned}
g_u^{(n)}(x, y) &= \sum_{k=0}^{n} \left[\frac{n!}{k!\,(n-k)!}\, \hat{u}_x^k\, \hat{u}_y^{n-k}\, \frac{\partial^n\, g(x, y)}{\partial x^k\, \partial y^{n-k}}\right] \\
&= \mathbf{v}_s(\hat{\mathbf{u}}) \cdot \mathbf{b}_s,
\end{aligned}
\tag{22}
$$

where the elements of $\mathbf{b}_s$ are the nth-order separable derivatives (each a function of x and y) and the vector $\mathbf{v}_s$ depends on the derivative direction.

6

So any directional derivative can be expressed in terms of a basis set of separable derivatives. But this is not quite the same as steerability. We want to express any directional derivative in terms of a basis set of other *directional* derivatives. To do that, we simply make a change of basis.

Since we can write any directional derivative as a weighted sum of separable derivatives, we can take $n + 1$ directional derivatives in different directions and write them all in terms of the $n + 1$ separable derivatives:

$$\mathbf{b}_d = \mathbf{M}\,\mathbf{b}_s. \tag{23}$$

The notation can get kind of confusing here: each element of the "vector" $\mathbf{b}_d$ is one of the directional derivatives (each a function of x and y) and each element of the "vector" $\mathbf{b}_s$ is one of the separable derivatives (each a function of x and y). Each element of the matrix $\mathbf{M}$ is a number $\frac{n!}{k!\,(n-k)!}\hat{u}_x^k\hat{u}_y^{n-k}$, that depends on the direction $\hat{\mathbf{u}}$. Each row of $\mathbf{M}$ has a different $\hat{\mathbf{u}}$ and each column of $\mathbf{M}$ has a different value for k counting up from 0 to n.

Finally, we combine the above two equations to write any directional derivative in terms of this new (directional derivative) basis:

$$
\begin{aligned}
g_u^{(n)}(x, y) &= [\mathbf{v}_s^t(\hat{\mathbf{u}})\,\mathbf{M}^{-1}]\,\mathbf{b}_d \\
&= \mathbf{v}_d(\hat{\mathbf{u}}) \cdot \mathbf{b}_d,
\end{aligned}
\tag{24}
$$

where $\mathbf{v}_d(\hat{\mathbf{u}})$ are the interpolation functions and $\mathbf{b}_d$ is the basis set.

To review, this section showed that n^{th}-order directional derivatives of a 2d Gaussian are steerable with a basis set of size $n + 1$.

Although the directional derivatives provide a useful example, the principle of steerability is *not* limited to such functions. The notion of steerability can be generalized to include any polar-separable function with a band-limited angular component, and arbitrary radial component [10, 11].

In addition, steerability is not limited to rotation. Hel-Or and Teo [4] have developed a theory for designing basis sets that are "steerable" with respect to a wide variety of spatial transformations including affine transformations.

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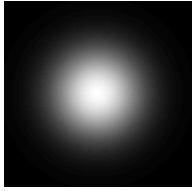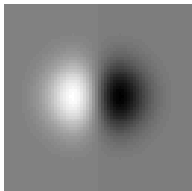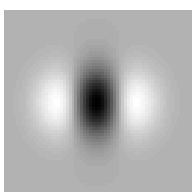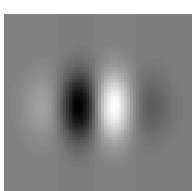
Filter	Formula
	$g(x,y) = e^{-(x^2+y^2)/2}$
	$g_x(x,y) = -xg(x,y)$
	$g_{xx}(x,y) = [x^2 - 1]\, g(x,y)$
	$g_{xxx}(x,y) = [3x - x^3]\, g(x,y)$

Figure 2: Steerability of higher-order derivatives. Illustrated is a 2d Gaussian and its first-through third-order directional derivatives.

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