Sporadic chaos in space-time dynamical processes

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We characterize a class of strongly intermittent processes in space and time as sporadic chaos, for which the dynamical entropy (a measure of randomness or information content) per unit time is not extensive and exhibits fractal scaling with the system size. Theoretical arguments are presented to show that directed percolation processes at the criticality are sporadically chaotic in that sense. For deterministic sporadic chaos, many positive Lyapunov exponents exist but the information generation rate per unit time and unit space volume is zero. An analysis of the Lyapunov exponent spectrum is carried out on the Chaté-Manneville coupled map lattice model at the transition point from rest to space-time chaos. The scaling exponents for the dynamical entropy per unit time and the attractor's Lyapunov dimension are estimated. Furthermore, the accumulation of exponents near zero gives rise to a divergence of the Lyapunov exponent distribution density at zero.

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I. INTRODUCTION

Many dynamical systems in nature display complex behaviors where random variations and coherent patterns are interspersed in space and time [1]. Imagine such a strongly intermittent system. Suppose that if one looks at a spatially localized observable its time evolution is so dominated by regular behavior or simply by quiescence that the chaotic activity occurs only on a fractal subset of time with zero measure. On the other hand, if one looks at the spatial pattern at a given time, active fluctuations are also confined to a fractal subset, embedded in an otherwise well ordered or silent space. How can one quantitatively characterize the space-time dynamics of such systems? In what precise sense can they be identified as chaotic or "at the border of chaos"?

In this work we propose to examine these questions in terms of the basic concepts of chaos theory, namely, the Lyapunov exponents and the Kolmogorov-Sinai entropy $h_{\rm KS}$ [2]. Chaotic time evolution of a deterministic differentiable system has at least one positive Lyapunov exponent λ and displays exponential sensitivity to initial conditions. Because of the Pesin's equality (which we assume to hold in general for attractors)

$$h_{\rm KS}=\sum_{\lambda_i>0}\lambda_i \ , \tag{1.1}$$

$h_{\rm KS}$ is also strictly positive. The dynamical entropy $h_{\rm KS}$ measures the information production rate per unit time by the process. It can be expressed as the limit

$$h_{\rm KS}=\lim_{T\to\infty}\frac{1}{T}H(T) \ , \tag{1.2}$$

where $H(T)$ is the entropy ("information content") of the process over a time interval T. Furthermore, if λ_i are labeled in a decreasing order, then the Lyapunov dimension D_L of the attractor is

$$D_L=k+\frac{\sum_{j=1}^{k}\lambda_j}{|\lambda_{k+1}|} \ , \quad k=\max\left\{m\ \middle|\ \sum_{j=1}^{m}\lambda_j\geq 0\right\} . \tag{1.3}$$

On the other hand, it has been shown [3,4] that strong temporal intermittency can display *sporadic* chaos, in the sense that

$$H(T)\underset{T\to\infty}{\sim}\begin{cases}T^{\alpha} & (0<\alpha<1)\\ T/(\ln T)^{\tilde\alpha} & (\tilde\alpha>0) \ ,\end{cases} \tag{1.4}$$

and nearby trajectories diverge as $\delta x_t\sim\exp(ct^{\alpha})\delta x_0$, or $\delta x_t\sim\exp[ct/(\ln t)^{\tilde\alpha}]\delta x_0$, with $c>0$. For sporadic time evolution, no positive Lyapunov exponent exists and the entropy per unit time is zero. Examples of this kind include the Pomeau-Manneville dynamics at the criticality [5] and a quantum spin system [6]. This phenomenon can also be realized by purely stochastic processes, in which case the Shannon entropy for a time interval T behaves as Eq. (1.4). In fact, Mandelbrot first investigated such sporadic random functions and pointed out the unusual behavior of Shannon entropy in the context of information theory of communication [7,8]. More recently, it was shown that the Lévy motion (a model of anomalous diffusion) [9] can be sporadic in the same sense [10].

Now consider a highly intermittent dynamical system with many active elements (or degrees of freedom) in a large spatial extension L (of volume $V=L^d$, d being the spatial dimensionality). One can easily construct peculiar examples for which the *space-time* dynamical entropy $H(V,T)$ scales with V and T anomalously. For instance, if independent critical Pomeau-Manneville maps are distributed on a discrete lattice or on a fractal sublattice of dimension d_f, one can have, respectively,

$$H(V,T)\sim\begin{cases}T^{\alpha}V\\ T^{\alpha}V^{d_f/d}\end{cases} \tag{1.5}$$

for $0<\alpha$ and $d_f/d<1$. The largest Lyapunov exponent is zero and the system is *temporally* sporadic.

It has been suggested that certain "self-organized critical" (SOC) systems [11,12] are characterized by the absence of positive Lyapunov exponents. This, however, has been shown numerically to be untrue in a SOC earthquake model [13] and is likely so for most SOC processes and for extended dynamical systems in general. In the case of fully developed fluid turbulence, which has many positive Lyapunov exponents, Ruelle has investigated the *spectrum* of Lyapunov exponents and suggested that strong intermittency might be manifested by an infinite distribution density of Lyapunov exponents at $\lambda=0$ [14]. Evidence for such a phenomenon has been seen in *ad hoc* shell models of scalar turbulence [15].

In the present work, we would like to focus on the *scaling* of the dynamical entropy per unit time $h_{\mathrm{KS}}(V)$ as function of the size of the system V. The existence of positive Lyapunov exponents is equivalent to the strict positiveness of $h_{\mathrm{KS}}(V)$, according to Eq. (1.1). We shall refer to *space-time sporadic chaos* a large chaotic system for which the dynamical entropy $h_{\mathrm{KS}}(V)$ is not extensive:

$$h_{\mathrm{KS}}(V) \underset{V\to\infty}{\sim} V^{\alpha} \quad (0<\alpha<1)$$

or (1.6)

$$V/(\ln V)^{\tilde{\alpha}} \quad (\tilde{\alpha}>0)\ .$$

For instance, how $h_{\mathrm{KS}}(V)$ of fluid turbulence scales with V is an open question.

In terms of the dynamical entropy, our approach can be applied to deterministic systems as well as stochastic processes. In Sec. II we shall provide arguments to show that the sporadic behavior of Eq. (1.6) is a general property of the stochastic directed percolation (DP) processes at the criticality. Since many large nonequilibrium dynamical systems displaying a continuous phase transition belong to the DP universality class, the notion of sporadic chaos could provide a general measure of dynamical chaos for strongly intermittent processes in space and time. In Sec. III we shall consider a well known *deterministic* model of space-time intermittency—the coupled map lattice of Chaté and Manneville—and show by numerical simulations that the model at the critical point of transition from resting behavior to space-time chaos is characterized by sporadic chaos in the sense of Eq. (1.6) as well as by a divergence of distribution density of Lyapunov exponents at $\lambda=0$ (which we shall refer to as strong intermittency in the sense of Ruelle). The possibility that SOC systems may display sporadic chaos in a *robust* way is briefly discussed in Sec. IV.

II. DIRECTED PERCOLATION

Many model systems of large extended nonequilibrium processes display a continuous (second-order) phase transition from a time independent ("absorbing") state to a spatiotemporally random state (the active state). Examples arise from diverse fields, including chemical reaction-diffusion systems [16], heterogeneous catalytic surface [17], evolution of galaxies [18], and epidemic contact process [19]. It turns out that all these models belong to the same universality class DP [20]. Indeed, the conjecture has been put forth that DP is the generic critical behavior of models with a single order parameter, which exhibit a continuous phase transition to a unique absorbing state [21–23].

We would like to propose here that for any dynamical process of the DP universality class, the critical behavior at the phase transition is characterized by space-time sporadic chaos in the sense of Eq. (1.6). The plausibility of this hypothesis can be seen by a heuristic argument based on the self-affine fractal properties of the critical DP clusters. Consider a DP process (in discrete space-time) at the criticality, the oriented direction being viewed as time. The asymptotic activity pattern of the system is dominated by an infinite space-time cluster. According to the scaling theory [20], the active sites at a given time forms a fractal that scales with the system size L as

$$N(L)\sim L^{D_\perp}\ , \quad D_\perp = d-\beta/\nu_\perp \tag{2.1}$$

and the temporal activity at a fixed site forms a fractal that scales with the total time of observation T as

$$N(T)\sim T^{D_\parallel}\ , \quad D_\parallel = 1-\beta/\nu_\parallel\ , \tag{2.2}$$

where d is the dimensionality of the physical space, β is the critical exponent for the stationary active site density, and $\nu_\perp$ and $\nu_\parallel$ are the critical exponents for the correlation lengths in space and time, respectively. The numerical values of these exponents are summarized in Table I.

TABLE I. Critical exponents β, $\nu_\parallel$, and $\nu_\perp$ for directed percolation process of spatial dimension d. Numerical values are obtained from Essam, Guttmann, and De'Bell [24] (low density series expansion) for $d=1$, Grassberger [25] (Monte Carlo simulation) combined with De'Bell and Essam [26] (series expansion) for $d=2$, Janssen [21] (ϵ expansion around the upper critical dimension $d_c=4$) for $d=3,4$. $D_\parallel$ is the fractal dimension of temporal activity at a given spatial location and $D_\perp$ is the fractal dimension of spatial activity for a fixed time. The space-time activity pattern forms a self-affine fractal of dimension $D_F=1+D_\perp$. The dynamical entropy per unit time scales with the system volume with the sporadicity exponent α.

d	β	$\nu_\parallel$	$\nu_\perp$	$D_\parallel=1-\beta/\nu_\parallel$	$D_\perp=d-\beta/\nu_\perp$	$\alpha=1-\beta/(\nu_\perp d)$
1	0.2765±0.0003	1.7339±0.0003	1.0969±0.0003	0.8405±0.0002	0.7479±0.0003	0.7479±0.0003
2	0.586±0.015	1.286±0.005	0.729±0.008	0.544±0.014	1.198±0.03	0.598±0.015
3	0.82	1.10	0.58	0.25	1.59	0.53
4	1	1	$\frac{1}{2}$	0	2	$\frac{1}{2}$

Now, the space-time activity forms a self-affine cluster, of which the fractal dimension D_F has been carefully considered in [27]. Numerical results using the box-counting method lend support to the formula

$$D_F = d + 1 - \beta/\nu_\perp , \tag{2.3}$$

which is corroborated by another independent Monte Carlo study [28]. Indeed, this expression is expected considering that all the cuts not in the time direction are generic [29], with fractal dimension $D_\perp$. Thus $D_F = 1 + D_\perp$.

Let us find some upper bounds for the space-time dynamical entropy $H(V,T)$. If the temporal correlations are ignored, then

$$H_s(V,T) = TH_s(V) , \quad H_s(V) = -\sum_i P_i^s \log_2 P_i^s , \tag{2.4}$$

where $H_s(V)$ is the spatial entropy, P_i^s is the stationary probability for a spatial configuration labeled by the index i, and the sum is over all possible configurations.

Furthermore, if all the spatiotemporal correlations are neglected and if in the asymptotic state the probability p for any space-time site being active is the same, then

$$\begin{gathered} H_0(V,T) = VTh_0 , \\ h_0 = -p \log_2 p - (1-p) \log_2 (1-p) . \end{gathered} \tag{2.5}$$

We have

$$H(V,T) \le H_s(V,T) \le H_0(V,T) . \tag{2.6}$$

It is reasonable to assume that all three quantities scale with V and T with the same exponents and we would like to evaluate these scaling laws for $H_0(V,T)$ and $H_s(V,T)$.

Equation (2.3) implies that the number of active sites in a space-time domain $V \times T$ would scale on average like $N(V,T) \sim TV^{D_\perp/d}$; the mean probability p for any space-time site being active is then

$$p \sim \frac{N(V,T)}{TV} \sim V^{-\beta/(\nu_\perp d)} \underset{V\to\infty}{\longrightarrow} 0 . \tag{2.7}$$

Ignoring correlations and assuming equidistribution, therefore,

$$H_0(V,T) = VTh_0 \sim TV^{1-\beta/(\nu_\perp d)} O(\log_2 V) \tag{2.8}$$

and the entropy per unit time is

$$H_0(V) = \lim_{T\to\infty} \frac{1}{T} H_0(V,T) \sim V^\alpha O(\log_2 V) , \quad \alpha = 1 - \beta/(\nu_\perp d) . \tag{2.9}$$

On the other hand, Kinzel has analyzed the scaling of the stationary spatial entropy $H_s(V)$ in a one-dimensional probabilistic cellular automaton at a DP criticality and using the finite size scaling technique showed that $H_s(V) \sim V^{1-\beta/\nu_\perp}$ [30]. Generalizing his reasoning to the d-dimensional case, then

$$H_s(V,T) \sim TV^\alpha , \quad \alpha = 1 - \beta/(\nu_\perp d) , \tag{2.10}$$

which agrees with the scaling of $H_0(V,T)$ except for the logarithmic factor. These results suggest that the critical DP process exhibits sporadic chaos in the sense that the dynamical entropy per unit time is positive, but it scales sublinearly with the system size [cf. Eq. (1.6)]. The *sporadicity exponent* $\alpha = 1 - \beta/(\nu_\perp d)$. The numerical values of α for different dimensionalities are listed in Table I.

That the dynamical entropy per unit time $H(V) = \lim_{T\to\infty} H(V,T)/T$ is strictly positive means that the critical DP dynamics is chaotic. A finite subsystem generates a finite amount of information per time step, with a rate that increases with the subsystem's size monotonically albeit sublinearly. This conclusion is not obvious; actually an opposite impression might be obtained if one looks at space and time disjointly rather than as a whole. Suppose that one is limited to monitor the time evolution at a fixed site and let us ask what the information generation rate per unit time h_t by that temporal signal is. According to Eq. (2.2), the probability for a given site being active at any particular time instant is $p_t \sim T^{D_\parallel}/T$. Hence

$$h_t \sim -p_t \log_2 p_t - (1-p_t) \log_2 (1-p_t) \sim T^{-\beta/\nu_\parallel} \underset{T\to\infty}{\longrightarrow} 0 . \tag{2.11}$$

Similarly, at a fixed time the probability for any site being active is $p_s \sim V^{D_\perp/d}/V$ [Eq. (2.1)]. Hence the information generation rate per unit space volume is

$$\begin{aligned} h_s &\sim -p_s \log_2 p_s - (1-p_s) \log_2 (1-p_s) \\ &\sim V^{-\beta/(\nu_\perp d)} \underset{V\to\infty}{\longrightarrow} 0 . \end{aligned} \tag{2.12}$$

However, the space-time pattern is not a direct product of the time cut with the space cut and *collectively* the dynamical interactions between spatial units generates an amount of information per unit time that grows infinitely with the system size.

III. DETERMINISTIC CHATÉ-MANNEVILLE MODEL

The Chaté-Manneville coupled map lattice [31] is perhaps the best known *deterministic* model that exhibits a continuous phase transition from an absorbing quiescent state to a chaotic state and near the criticality displays strong space-time intermittency. Initial numerical results suggested that although the transition is DP-like, it does not belong to the DP universality class. A later study using much larger lattice size and longer time, however, yielded critical exponent values that are compatible with the DP prediction [32].

For deterministic systems the dynamical entropy is connected with the Lyapunov exponents and here we would like to study the scaling properties of the entropy and the Lyapunov exponent spectrum in the critical Chaté-Manneville model. Let us consider a variant form of it [32], which in one dimension is described by

$$x_i^{t+1}=f\left[(1-\epsilon)x_i^t+\frac{\epsilon}{2}(x_{i-1}^t+x_{i+1}^t)\right],$$

$$f(x)=\begin{cases} rx & \text{if } 0<x\le\frac{1}{2} \\ r(1-x) & \text{if } \frac{1}{2}<x\le 1 \\ x & \text{if } x>1 , \end{cases} \tag{3.1}$$

with $i=1,2,\ldots,n$, n being the system size. The periodic boundary condition is used.

Let $r=3$ be fixed. The mapping dynamics $f(x)$ is chaotic (or active) for $0<x<1$ and stationary (or inactive) for $x>1$. It is readily seen that for weak coupling (small ϵ values), the long-term behavior of the lattice system is a steady state where every site is inactive. Note that there is a continuous degeneracy of such "absorbing states." It is known that as ϵ is increased to $\epsilon_c\simeq 0.359\,85$ a transition occurs, giving rise to spatiotemporal chaos for $\epsilon>\epsilon_c$ [31,32]. Chaté has recently analyzed the scaling properties of the behavior as $\epsilon\to\epsilon_c^+$, in terms of the spectrum of Lyapunov exponents and vectors [33]. In that work it is shown that as $\epsilon\to\epsilon_c^+$, the fraction ρ_{act} of active sites tends to zero, while the largest Lyapunov exponent remains finite. Indeed, many positive Lyapunov exponents seem to exist even at $\epsilon=\epsilon_c$.

Here we are concerned with the scaling of the Lyapunov spectrum as function of the lattice size n, at the criticality. The complete Lyapunov spectrum was computed at $\epsilon=0.359\,85$ using the method of Gram-Schmidt orthonormalization [34]. The dynamical entropy [$h_{\text{KS}}(n)$] and Lyapunov dimension [$D_L(n)$] are calculated according to Eqs. (1.1) and (1.3), respectively. Note that for a finite lattice size n, at the criticality the system will always converge to the absorbing state. Only in the limit of infinite lattice is the asymptotic dynamics nontrivial. In simulations, we started with a random initial condition [the initial state at each site was chosen from a uniform distribution on $(0,r/2)$]. After a long enough transient period of time but before the dynamics started to decay to the absorbing state, a time interval T was chosen, over which the Lyapunov spectrum was calculated. The Gram-Schmidt procedure requires a computation time that is proportional to n^3T. This cubic dependence on the lattice size makes very large-scale simulations unpractical at the present time. For instance, a calculation with $n=2048$ and $T=1000$ takes about 22.5 h of CPU time on a Cray Y-MP supercomputer. We have performed the calculations with $n=128$, 256, 512, 1024, and 2048; for each lattice size the Lyapunov spectrum was computed with a time interval $T=1000$ and with several random initial conditions. In fact, we are not aware of a Lyapunov spectrum analysis on such a large dynamical system being reported so far in the literature.

The simulation results clearly show that both the dynamical entropy $h_{\text{KS}}(n)$ and Lyapunov dimension $D_L(n)$ increase with lattice size n; hence the model at the criticality is chaotic with sensitivity to initial conditions. Figure 1 shows the log-log plots of $h_{\text{KS}}(n)$ and $D_L(n)$ with four different random initial conditions. In each case the linear regression method yields a statistical fit of straight line, with a slope $\alpha=0.71\pm0.06$ for h_{KS} and $\kappa=0.85\pm0.03$ for D_L, respectively. Thus we have

$$\begin{aligned} h_{\text{KS}}(n)&\sim n^{\alpha}, \quad \alpha=0.71\pm0.06 \\ D_L(n)&\sim n^{\kappa}, \quad \kappa=0.85\pm0.03 . \end{aligned} \tag{3.2}$$

The estimated value of the exponent α is compatible with the DP prediction (see Table I). By analogy with DP, we hypothesize that α is equal to the fractal dimension of the spatial set of active sites for fixed time. On the other hand, the exponent κ seems to be new and it is unknown whether it might be expressed in terms of other critical exponents of the system through a hyperscaling relation.

Therefore, both $h_{\text{KS}}(n)$ and $D_L(n)$ scale sublinearly with the system size and we conclude that the critical Chaté-Manneville model represents an example of space-time sporadic chaos in the sense of Eq. (1.6). Note that the variations of $h_{\text{KS}}(n)$ and $D_L(n)$ are greater with larger values of n. Indeed, it is quit conceivable that a quantity whose mean scales sublinearly may have variance that scales *anomalously* [3,4,10]. The study of fluc-

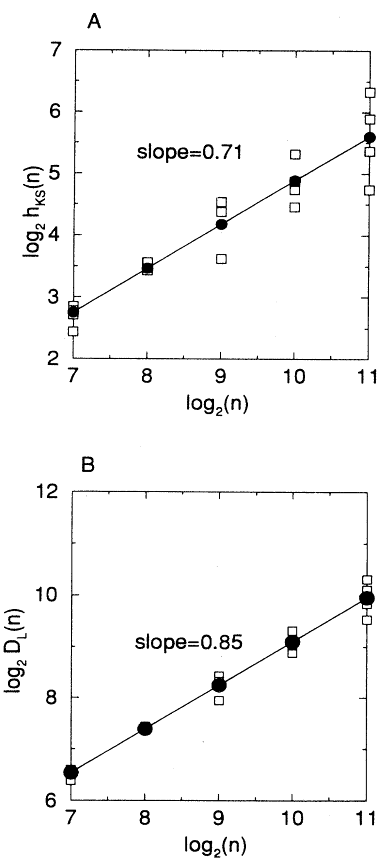

FIG. 1. The log-log plot of (a) the dynamical entropy $h_{\text{KS}}(n)$ and (b) the Lyapunov dimension $D_L(n)$ as function of the lattice size for the critical Chaté-Manneville model ($r=3$, $\epsilon=0.359\,85$). For each $n=2^k$, $k=7,\ldots,11$, four data points (open square) correspond to different random initial conditions. The linear regression (solid lines) of the raw data yields a scaling exponent $\alpha=0.71\pm0.06$ for $h_{\text{KS}}(n)$ and $\kappa=0.85\pm0.03$ for $D_L(n)$.

tuations is not included in the present work.

In Fig. 2(a) the Lyapunov spectrum as function of the lattice size is shown, where $\lambda_i = \lambda(x)$ are plotted in an increasing order with $x = i/n$. Note that the inverse $x = F(\lambda)$ is the distribution function for the exponents. Displayed in Fig. 2(a) is one of the four samples of which $h_{\mathrm{KS}}(n)$ and $D_L(n)$ are closest to the linear regression lines. One sees that, in agreement with Chaté [33], the largest Lyapunov exponent $\lambda_{\max} \simeq 0.43$ does not change significantly with n and the curves look continuous. This implies that in the limit $n \rightarrow \infty$, there will be a whole continuous range of positive Lyapunov exponents. On the other hand, for each fixed n, the area below the curve and for $\lambda \geq 0$ (shaded for $n = 2048$) is simply $(1/n)\sum_{\lambda_i > 0} \lambda_i = h_{\mathrm{KS}}(n)/n$, which clearly decreases with n. That $h_{\mathrm{KS}}(n)$ grows sublinearly with n implies that this area should vanish as $n \rightarrow \infty$.

Another essential feature of the Lyapunov spectrum is the existence of many near-zero exponents, the number of which increases with the lattice size. For $n = 2048$ an enlarged portion of the spectrum close to $\lambda = 0$ is shown in the inset of Fig. 2(a). This observation led us to surmise that in the limit of $n = \infty$, the distribution function $F(\lambda)$ has an infinite derivative at $\lambda = 0$. To assess this hypothesis we performed the averaged distribution *density* $\rho(\lambda) = F'(\lambda)$ for $n = 512$, 1024, and 2048 [Fig. 2(b)]. For a fixed n value, the distribution density for each sample was calculated as the smoothed derivative of $F(\lambda)$ with a bin $\Delta\lambda = 0.01$; then an average was made over the four samples. One sees in Fig. 2(b) that the three curves largely coincide with each other for $\lambda < 0$. For positive λ the density is a decreasing function of n (see inset). Near $\lambda = 0$, there is a peak that rapidly grows with n and seems to diverge as $n \rightarrow \infty$.

Let us assume that near $\lambda = 0-$, $F(\lambda) = F_c - a|\lambda|^{\mu}$ ($a > 0$) and try to extract an estimate for μ from our data with $n = 2048$. Let us call any Lyapunov mode "in the interior of the attractor" if it adds to the Lyapunov dimension. [According to Eq. (1.1), if λ_i are labeled in decreasing order, the mode i is in the interior of the attractor if $i < D_L$.] Let K_p (K_n) be the number of positive (negative) such modes; then $[D_L] = K_p + K_n$, where $[x]$ means the integer part of x. Now let the negative Lyapunov modes in the interior of the attractor be λ_j, $j = K_0+1, K_0+2, \ldots, K_0+K_n$, in *increasing* order; then $\lambda_{K_0+K_n+1}$ is the smallest positive Lyapunov exponent. Let us also determine a value K_c ($[K_c] = K_0 + K_n$) by linear extrapolation between $\lambda_{K_0+K_n}$ and $\lambda_{K_0+K_n+1}$, so that $\lambda_{K_c} = 0$. Then $(K_c - j)/n$ is the discrete approximation of $F_c - F(\lambda)$ for negative exponents in the interior of the attractor. For fixed $n = 2048$ and for each of the four samples, we plot in Fig. 3 $\log_{10}[(K_c - j)/n]$ as a function

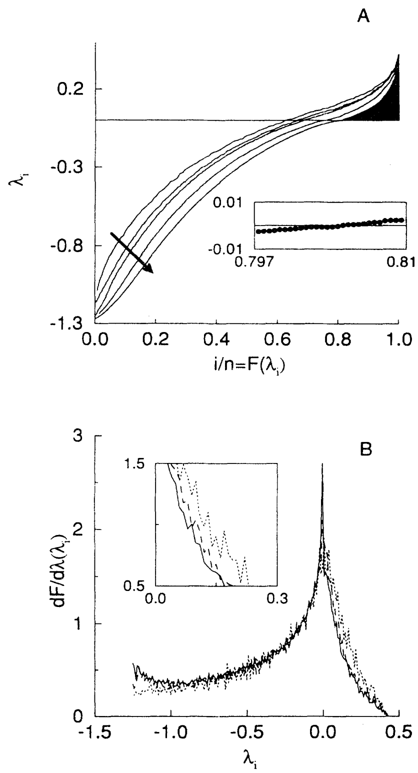

FIG. 2. (a) A sample Lyapunov exponent spectrum for $n = 128$, 256, 512, 1024, and 2048 (arrow indicates increasing n). The maximum Lyapunov exponent value ($\simeq 0.43$) does not vary significantly with n. The strong intermittency is manifested by the presence of many near-zero exponents (inset for $n = 2048$) and by the fact that the dynamical entropy per unit time and space $h_{\mathrm{KS}}(n)/n$ (shaded area shown for $n = 2048$) decreases to zero as $n \rightarrow \infty$. (b) The smoothed and averaged distribution density of Lyapunov exponents are shown with $n = 512$ (dotted), 1024 (dashed), and 2048 (solid). The three curves are essentially the same for $\lambda < 0$, while for $\lambda > 0$ the distribution density is a decreasing function of n (inset). Note a large peak at $\lambda = 0$, indicating that in the limit of infinite lattice $dF/d\lambda(\lambda = 0) = \infty$.

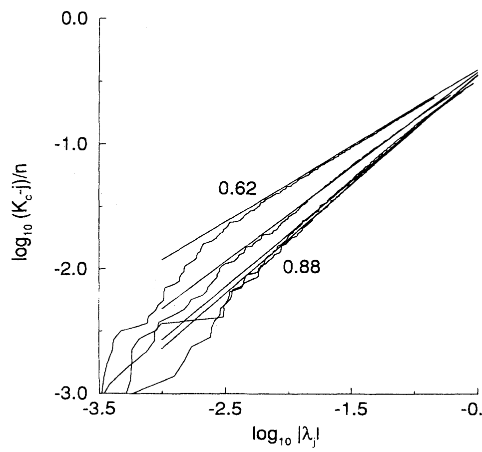

FIG. 3. For $n = 2048$, negative exponents in the interior of the attractor are labeled in increasing order and K_c is the linearly extrapolated value such that $\lambda_{K_c} = 0$. The log-log plot of $(K_c - j)/n$ versus λ_j yields a linear fit, except very close to $j = [K_c]$, for each of the four samples. The estimated slope varies from 0.61 to 0.88, with a mean of 0.78, which is less than 1. This implies that in the limit of infinite system, the distribution function of Lyapunov exponents $F(\lambda) \simeq 1 - a|\lambda|^{\mu}$, $\mu < 1$, for small negative λ values near zero.

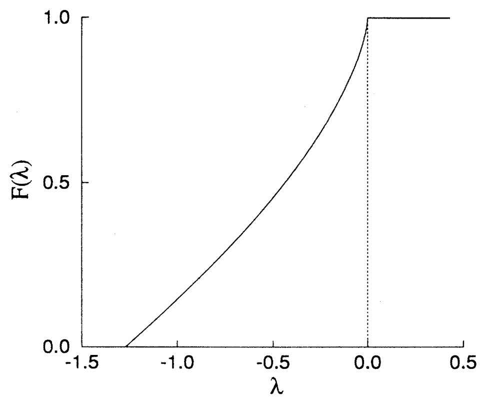

FIG. 4. Schematic drawing of the limiting distribution $F(\lambda)$ of Lyapunov exponents, characterized by a horizontal plateau for $\lambda \in [0,\lambda_{max}]$ and an infinite derivative for $\lambda \to 0-$.

of $\log_{10}(|\lambda_j|)$, $j=K_0+1, K_0+2, \ldots, K_0+K_n$. One sees that a large portion of each curve is linear except near $j=K_0+K_n$, where errors may occur due to the limited accuracy (10^{-4}) with which the critical value K_c was determined and due to the finite system size [note that the discussion in the preceding paragraph implies that $\lim_{n\to\infty} K_c(n)/n = F_c = 1$]. The slope obtained from linear regression varies from 0.61 to 0.88, with a mean $\mu \simeq 0.78$. Hence $\mu < 1$; consequently $F'(\lambda=0)=\infty$. Combining the above results we conclude that in the limit of infinite lattice, the distribution function $F(\lambda)$ of Lyapunov exponents is continuous, with a constant plateau at $F(\lambda)=1$ for $\lambda \geq 0$, and its derivative diverges at $\lambda=0$ as the latter is approached from the negative side (see Fig. 4 for a schematic illustration).

IV. DISCUSSIONS

An intriguing question is whether sporadic chaos can be a robust phenomenon that persists under small changes of the underlying system such as tuning a control parameter. We have argued above that sporadic chaos is realized by critical directed percolation processes. One can then ask whether a large nonequilibrium system can be driven by its intrinsic dynamics into such a criticality. Recently, Bak and Sneppen introduced a simple model of biological evolution [35] and showed that it self-organizes into a critical dynamics displaying strongly intermittent coevolutionary avalanches of all sizes ("punctuated equilibrium"). The space-time activity ("mutation") seems to behave similarly to the critical directed percolation [36,37]. If this is true, then it follows from the argument of Sec. II that the space-time dynamics of the Bak-Sneppen model should be sporadically chaotic; we hypothesize that its space-time dynamical entropy $H(V,T)$ is linear with T, but that the entropy per unit time scales with the system size sublinearly. Note that for discrete models the notion of Lyapunov exponents is not well established. The dynamical entropy, on the other hand, is applicable and may be used to quantify the degree of chaos in those systems.

It thus appears that sporadic chaos can indeed occur generically in large extended nonlinear dynamical systems. The notion introduced here may be used to describe quantitatively the strongly intermittent space-time activity in self-organized critical processes, which give rise to the fractal scaling of information content of structures and patterns in physics, biology, or linguistics [38,39].

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