Dynamical sporadicity and anomalous diffusion in the Lévy motion

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We discuss continuous-time sporadic dynamics that are intermediate between regular and random behaviors. A characterization of such processes is provided by a scale-dependent entropic quantity, and is applied to a model of Lévy motion introduced by Klafter, Blumen, and Shlesinger [Phys. Rev. A **35**, 3081 (1987)]. The study suggests that sporadicity may be a feature of some physical systems exhibiting anomalous diffusion.

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I. INTRODUCTION

When it was first introduced in the turbulence literature, the term *intermittency* denoted the remarkable non-Gaussian nature of small-scale fluctuations in the velocity field of a fluid at high Reynolds number [1]. Efforts to understand this phenomenon were one of the motivations which led to the recognition that *strongly* intermittent structures can be identifiable with geometrical objects of fractional dimension, or fractals [2].

As far as *temporal* intermittency is concerned, there are two slightly different viewpoints. On the one hand, attention may be focused on certain special events which occur rarely in time, but which are nevertheless critical in determining physical properties of a system. On the other hand, when a dynamical evolution consists of both quiescent (or regular) and active (or chaotic) phases which alternate temporally in an interspersed way, one may wish to quantify the degree of the chaoticity of the process *as a whole.* A well-known example of such dynamical intermittency is the deterministic map of Pomeau and Manneville [3]

$$x_{n+1}=x_n+cx^z \pmod 1, \quad z>1, \quad c>0 \tag{1.1}$$

at the transition point from periodic oscillation to chaos. Note that, though dynamical systems are given by differential equations or iterations such as Eq. (1.1), via symbolic dynamics [4] the phase space can be partitioned into discrete cells, thereby trajectories are coded by strings of integers or symbols. In other words, deterministic systems with continuous variables correspond to stochastic processes with *discrete* states and time.

In Ref. [5] a criterion is given according to which *sporadic* dynamics is defined as intermediate between the regular and random behaviors, in low-dimensional deterministic systems or discrete stochastic processes. In the present work we propose to consider stochastic processes which are *continuous* both in observables and in time, following the approach of Ref. [5]. This generalization seems of interest, since "large" dynamical systems are likely to be connected to continuous stochastic processes. This is a basic assumption, for instance, of the statistical theory of full developed fluid turbulence, which involves many active degrees of freedom [1]. In this paper we shall be restricted only with dynamical systems of a few degrees of freedom but in a large spatial extension, such as the motion of a tracer particle immersed in a fluid. Transport phenomena such as diffusion of scalar tracers have been demonstrated experimentally in an array of convection rolls or vortices [6]. In theoretical models like the Lorentz gas (or extended Sinai billiard) [7], deterministic chaos gives rise to a Brownian motion in real space [7,8] and the diffusion coefficient has been expressed in terms of the ergodic quantities of chaotic dynamics of the particle [9]. Perhaps more surprisingly, it was discovered [10] that diffusion of a scalar tracer in a fluid of small viscosity admits *anomalous* regimes, when the mean-square displacement does not grow linearly with time, $\langle \mathbf{r}^2(t)\rangle \sim t^\eta$, $\eta\neq 1$. Now, anomalous diffusion implies non-Gaussian fluctuations, and in some cases the particle can display Lévy rather than Brownian motion [11]. Furthermore, so far as random diffusion is related to an underlying chaotic dynamics, in an anomalous transport regime certain nonchaotic mechanisms may coexist with a chaotic one, inducing correlated effects and interesting intermittent phenomena. We shall show that the velocity of a particle undergoing a Lévy motion may be sporadic rather than random, in the sense specified below.

The content of this paper is outlined as follows. In Sec. II we recall the characterization of sporadic dynamics as proposed in Ref. [5], and discuss a natural way to extend it to continuous stochastic processes. In Sec. III, a continuous-time model of Lévy motion proposed by Klafter, Blumen, and Shlesinger (KBS) [12] is presented, and is analyzed using the theory of regenerative phenomena developed by Feller and Kingman [13,14]. Non-Gaussian fluctuations are discussed, and are shown to be described by the Lévy stable distributions. By the same token we obtain the asymptotic forms for the mean-square displacement $\langle \mathbf{r}^2(t)\rangle \sim t^\eta$. The four transport regimes which are found in Ref. [12] are thereby recovered in a more direct way, and interpreted in terms of the formation of fractals in space and/or time.

Section IV will be concerned with the velocity $\mathbf{v}(t)$ of a Lévy particle. We show that the time-invariant probability density for the velocity may or may not be normalizable. The autocorrelation function and the power spectrum are calculated, with the emphasis on the necessity to use conditional probabilities when the invariant probability density is not normalizable. From the velocity we derive again the long-term behaviors for $\langle \mathbf{r}^2(t)\rangle$. Finally, we show that sporadicity as defined in Sec. II is realized by the velocity $\mathbf{v}(t)$ of a Lévy particle, in a subclass of the KBS models. Some possible applications are mentioned in Sec V.

II. SPORADIC DYNAMICS AND ENTROPY

A. Intermittency and sporadicity

Intermittency is observable in a turbulent flow by quantities such as high-pass-filtered velocity or velocity derivatives. Those signals may display long quiescent ("nonturbulent") time intervals, interrupted by turbulent bursts. "Quiescent" here may mean for an observable to remain constant or below a prescribed threshold, and a "burst" the occurrence of a fluctuation exceeding the threshold. In order to measure the intermittency of a turbulent flow, Townsend [15] introduced a number $\tilde{\gamma}$ as the mean fraction of time a given signal is turbulent. Let $U(t)$ be the integrated turbulent time during $(0,t)$. Then

$$\tilde{\gamma}=\lim_{t\to\infty}\frac{\langle U(t)\rangle}{t} . \tag{2.1}$$

This number has been measured in boundary-layer experiments [16].

Partly motivated by the interest in fluid turbulence, in Ref. [17] Mandelbrot proposed a general classification of stationary stochastic processes exhibiting long quiescent time intervals. Let $\Phi(t)$ be the probability that an observable $X(t)$ is *not* quiescent (e.g., does not remain constant or stay below a prescribed threshold) over a time interval $(0,t)$. Then, $X(t)$ is said to be *finitely (infinitely)* intermittent if $\lim_{t\to\infty}\Phi(t)<\infty$ $(=\infty)$. Infinite intermittency was also called *sporadic behavior.* Let $\Phi(t)$ be written in an integral form

$$\Phi(t)=\Phi(0)+\int_0^t\phi(t')dt' . \tag{2.2}$$

Then, the sporadic behavior in the sense of Mandelbrot corresponds to the non-normalizability of the probability distribution $\phi(t)$.

In order to see the meaning of Mandelbrot's definition, one may interpret $\phi(t)$ as the probability for any given instant to be exactly at time t prior to the occurrence of an event. Then, its integral $\int_0^t\phi(t')dt'$ is the probability that the initial instant 0 is at a time shorter than or equal to t before an event occurs, which is the same as the probability $\Phi(t)$ that the time interval $(0,t)$ is not devoid of any event. Moreover, let us suppose that the events are regenerative. That is, once an event has occurred all the memory of the past is lost. For an intermittent signal from turbulent motion, for instance, though time correlation might persist during a long nonturbulent phase, it is expected to be destroyed by chaotic dynamics during a turbulent burst. In this case, one can write

$$\phi(t)=\phi(t+dt)+\phi(0)\varphi(t)dt , \tag{2.3}$$

where $\varphi(t)dt$ is the probability that the time lapse between two consecutive events is between t and $t+dt$. Equation (2.3) yields

$$\frac{d\phi(t)}{dt}=-\phi(0)\varphi(t)$$

or (2.4)

$$\phi(t)=\phi(0)\left[1-\int_0^t\varphi(t')dt'\right] .$$

Moreover, the quantity $\int_0^t t'\varphi(t')dt'$ tends to the first moment of $\varphi(t)$ as $t\to\infty$, i.e., the mean quiescent phase interval $\tau_1=\int_0^\infty t\varphi(t)dt$. Integrating by parts and using Eq. (2.4) one obtains

$$\int_0^t t'\varphi(t')dt'=\int_0^t\left[\int_0^t\varphi(t'')dt''-\int_0^{t'}\varphi(t'')dt''\right]dt' \underset{t\to\infty}{\sim} \frac{1}{\phi(0)}\int_0^t\phi(t')dt' . \tag{2.5}$$

Equation (2.5) asserts that $\phi(t)$ is not normalizable if and only if $\varphi(t)$ decays so slowly so that $\tau_1=\infty$. This is when no intrinsic time scale exists, and *fractal time* is expected to occur [18]. Indeed, as we shall see later, for a regenerative event $\tau_1=\infty$ generally implies that its $\langle U(t)\rangle$ grows nonlinearly with time, $\langle U(t)\rangle\sim t^\alpha$, where α is a fractal dimension for $U(t)$ $(0<\alpha<1)$. As a consequence, the intermittency factor $\tilde{\gamma}=0$ [cf. Eq. (2.1)].

One emphasizes that in the Mandelbrot sense fractal time corresponds to an extreme case of *infinite* intermittency. Non-normalizable probability densities do not pose mathematical difficulties if the theory of conditional probability of Rényi [19] is applied. For instance, average quantities over a time interval $(0,t)$, like correlation function or power spectrum, ought to be evaluated *conditioned* by the probability $\Phi(t)$ that *some* event must have occurred during $(0,t)$ (Ref. [17] and see below).

One may ask if a criterion for the sporadic behavior could be provided, without reference to such (somewhat arbitrary) terms as threshold, quiescent phase, or event. In the light of recent progress in the ergodic theory of dynamical systems, the notion of regular and chaotic behaviors became precise in terms of the Kolmogorov-Sinai entropy h_{KS} and the Lyapunov exponents [20], and of the algorithmic complexity of Kolmogorov and Chaitin [21]. A specific characterization of dynamical sporadicity, therefore, may be expressed in terms of such concepts.

B. Dynamical entropy

Suppose a low-dimensional dynamical system admits a generating partition of the phase space, so that a trajectory is represented by $S=(s_0s_1\cdots s_{n-1}\cdots)$, where s_i are integers or symbols of the partition. Then, S_n of length n is said to be *sporadic* [5], if its Kolmogorov-Chaitin algorithmic complexity $K(S_n)$ behaves as

$$K(S_n)\sim n^{\nu_0}(\ln n)^{\nu_1}\ , \tag{2.6}$$

where $\nu_0<1$, or $\nu_0=1$ and $\nu_1<0$. The regular ($\nu_0=0$) and random ($\nu_0=1$, $\nu_1=0$) cases are at the two extremities of the spectrum. It has been shown that the Pomeau-Manneville intermittent map Eq. (1.1) is sporadic for $2\le z$, with $\nu_0=1/(z-1)$. Sporadicity is realized when the random events occur rarely in time, often in a clustering manner. For $0<\nu_0<1$, they are restricted to a fractal-like subset in time, though the time here is discrete. The seemingly marginal case with $\nu_0=1$ and $\nu_1<0$ can in fact happen commonly, e.g., in Eq. (1.1) for $z=2$.

A relationship between $K(S_n)$ and the dynamical entropy asserts [4] that for almost all trajectories S_n,

$$\lim_{n\to\infty}K(S_n)/n=\lim_{n\to\infty}H_n/n=h_{\rm KS}\ , \tag{2.7}$$

where H_n is the Shannon-like entropy

$$H_n=-\sum_{S_n}\rho(S_n)\ln\rho(S_n)=\langle -\ln\rho(S_n)\rangle\ , \tag{2.8}$$

with ρ being an invariant measure of the system. The Kolmogorov-Sinai entropy $h_{\rm KS}$ is the rate of information creation per unit time. Thus, for a chaotic attractor with $\nu_0=1$ and $\nu_1=0$, a positive $h_{\rm KS}$ is present which, according to a theorem of Pesin [22]

$$h_{\rm KS}=\sum(\text{positive Lyapunov exponents}) \tag{2.9}$$

implies the existence of at least one positive Lyapunov characteristic exponent, and hence the system displays an exponential instability of trajectories.

From Eqs. (2.7) and (2.8), one expects that for a sporadic system, the scaling of the Shannon-like entropy H_n as a function of n behaves similarly as Eq. (2.6) [23]. As a consequence, the entropy *per unit time* will be zero, $h_{\rm KS}=0$. Besides, the dynamics may produce a *stretched* exponential instability, in the sense that along certain directions of the phase space, neighboring trajectories diverge as

$$\delta x(n)\approx\delta x(0)\exp[cn^{\nu_0}(\ln n)^{\nu_1}],\quad c>0 \tag{2.10}$$

in which case one may speak of a *sporadic Lyapunov mode,* with a zero Lyapunov exponent (defined as $\lambda=\lim_{n\to\infty}(1/n)\ln[\delta x(n)/\delta x(0)]=0$). Indeed, the Pomeau-Manneville system may be properly viewed as a description of a sporadic mode. Note that a sporadic mode is distinct from a regular (e.g., periodic) mode, for which one expects only a polynomial growth of perturbations

$$\delta x(n)\sim\delta x(0)+cn^{\nu},\quad \nu,c>0 \tag{2.11}$$

which again implies $\lambda=0$.

Also, for a multidimensional system, the presence of a sporadic mode does not imply necessarily a sporadic behavior of the entropy H_n. Chaotic modes with positive Lyapunov exponents which may be simultaneously present would imply a nonvanishing Kolmogorov-Sinai entropy.

C. ϵ entropy

For a genuinely continuous stochastic process, in contrast to a discrete one, since an exact knowledge of an analog random signal of time length T would require an infinite amount of information, an adequate entropic quantity is the amount of information production *within a given finite precision* ϵ, $H(\epsilon,T)$. This has been made rigorous by Kolmogorov and Shannon [24], in the context of the theory of information transmission. The ϵ entropy $h(\epsilon)=\lim_{T\to\infty}H(\epsilon,T)/T$ is the corresponding rate per unit time. In practice, ϵ may denote the precision limit by which analog signals are digitalized, and $h(\epsilon)$ is the corresponding Shannon entropy per unit time, for the digitalized signals. The asymptotic behaviors of $h(\epsilon)$ as ϵ goes to 0 have been recently used to describe and classify diverse dynamical systems [25].

Let us give two simple examples. The first example is a d-dimensional uniform distribution on the cube $[0,1]^d$. The d-dimensional vector output, digitalized by a resolution of ϵ, will have ϵ^{-d} possible values, with equal probability $\Pr(\epsilon)=\epsilon^d$. Hence a Shannon entropy of this coarse-grained random source will behave as

$$h(\epsilon)=-\ln\Pr(\epsilon)=d\ln(1/\epsilon)\ . \tag{2.12}$$

Indeed, a typical asymptotic form of $h(\epsilon)$ for a random function in a finite-dimensional space is $\ln(1/\epsilon)$ [24].

Our second example is an Ornstein-Uhlenbeck process which describes the velocity $v(t)$ of a Brownian particle undergoing a normal diffusion. The random function $v(t)$ obeys the Langevin equation

$$\frac{dv(t)}{dt}=-\lambda v+\xi(t)\ , \tag{2.13}$$

where $\xi(t)$ is a white noise with covariance $\langle\xi(t)\xi(t')\rangle=\delta(t-t')\sigma^2$. The transition probability is

$$p(v;t|v_0)=\frac{1}{(2\pi b\sigma^2)^{1/2}}\exp\left[-\frac{(v-v_0e^{-\lambda t})^2}{2b\sigma^2}\right]\ , \tag{2.14}$$

where $b=(1-e^{-2\lambda t})/2\lambda$. A simple reasoning to derive the asymptotic form of $H(\epsilon,T)$ for $v(t)$ is as follows. Consider a time window $[0,T]$, and let $t=k\tau$, $k=1,2,3,\ldots,n$ (so that the time is coarse grained by τ). According to Eq. (2.14), the transition from $v(k\tau)$ to $v((k+1)\tau)$ has a Gaussian distribution, with a mean $\langle v((k+1)\tau)|v(k\tau)\rangle\simeq v(k\tau)(1-\lambda\tau)$, and a variance $\mathrm{var}(v((1+k)\tau)|v(k\tau))\simeq\sigma^2\tau$. Therefore, if the standard deviation $\epsilon=\sigma\sqrt{\tau}$ is chosen as a resolution limit for $v(k\tau)$, at each time step the coarse-grained output will have a few highly probable values (e.g., 0, $+\epsilon$ or $-\epsilon$ from the mean value), and only a finite amount of information is generated. The entropic quantity $H(\epsilon,T)$ for the discretized $v(t)$ of n time steps can be shown to be a finite constant multiplied by n [25]. With $n=T/\tau=T\sigma^2/\epsilon^2$, we conclude that

$$H(\epsilon,T)\sim\frac{\sigma^2 T}{\epsilon^2}\ . \tag{2.15}$$

Since $H(\epsilon,T)$ is proportional to T, an ϵ entropy per unit time is positive, $h(\epsilon)=\lim_{T\to\infty}H(\epsilon,T)/T$, which behaves as $1/\epsilon^2$ for small ϵ. This divergence is obviously much faster than $\ln(1/\epsilon)$ as in the finite-dimensional cases.

By analogy with the discrete time case, we shall say that a continuous dynamical process is sporadic if its

$$H(\epsilon,T)\underset{T\to\infty}{\sim}T^{\nu_0}(\ln T)^{\nu_1}h(\epsilon)\ , \tag{2.16}$$

with $\nu_0<1$, or $\nu_0=1$ and $\nu_1<0$. In this case, the ϵ entropy per unit time for all fixed ϵ will be zero.

In what follows we shall show that sporadicity in the sense of Eq. (2.16) may be realized by the velocity of a Lévy motion rather than a Brownian one. This will be carried out on the KBS model introduced in Ref. [12].

III. LÉVY MOTION

A. KBS model

In the KBS model proposed in Ref. [12], a particle undergoes straight motion steps, interrupted by jumps. It is assumed that the memory is lost each time a jump has occurred, so that the process is entirely specified by the probability density $\Psi(\mathbf{r},t)$ for a single step of $\mathbf{r}$ in time t. $\Psi(\mathbf{r},t)$ has the following properties.

(i) $\int\int\Psi(\mathbf{r},t)dt\,d\mathbf{r}=1$.

(ii) $\Psi(\mathbf{r},t)=\psi(\mathbf{r})\delta(r-t^{\nu})$, thus the space and time are coupled by a δ function, and the velocity is constant during each step.

(iii) $\psi(\mathbf{r})$ depends only on r, and the orientation is uniformly distributed.

(iv) $\psi(\mathbf{r})\underset{r\to\infty}{\sim}r^{-\mu}$. Unusual behaviors are expected because of this long tail of $\psi(\mathbf{r})$. The distribution of the step length $\psi(r)$ is related to $\psi(\mathbf{r})$ by $\psi(\mathbf{r})d\mathbf{r}=\psi(r)dr$, thus one has $\psi(r)\sim r^{-\mu^*}$, where $\mu^*=\mu-d+1$ and d is the dimension of the physical space.

By a coordinate change one may rewrite $\Psi(\mathbf{r},t)$ as

$$\Psi(\mathbf{r},t)=\varphi(t)\delta(r-t^{\nu})\ , \tag{3.1}$$

where $\varphi(t)$ is the probability density of the interjump interval, satisfying

$$\varphi(t)\underset{t\to\infty}{\sim}at^{-(\alpha+1)},\quad F(t)=\int_t^{\infty}\varphi(t')dt'\sim At^{-\alpha}\ , \tag{3.2}$$

where $\alpha=\nu\mu^*-1$, A is a constant. We shall see that the statistical properties of the jump events are essentially dictated by the exponent α of the distribution $\varphi(t)$.

B. Regenerative event and fractal time

The particle's motion is basically determined by the jump events, which are governed by $\varphi(t)$. The jump events are *regenerative* in the sense that, if $p(t)$ denotes the probability that a jump occurs at time t, then the probability for jumps occurring at time instants $t_1,t_2,\ldots,t_n$ is

$$p(t_1,t_2,\ldots,t_n)=p(t_1)p(t_2-t_1)\cdots p(t_n-t_{n-1})\ . \tag{3.3}$$

There exists a well-developed theory of regenerative phenomena in the mathematics literature. The theory was first developed for discrete time processes by Feller [13], who used the term "recurrent phenomena," and later extended to the continuous time by Kingman [14]. These results provide us with powerful tools to analyze the regenerative events, in particular to establish a rigorous connection between non-Gaussian fluctuations with the Lévy stable distributions. The theory has been applied to the Pomeau-Manneville system [5,26], and to Hamiltonian chaotic systems [27].

According to Kingman's theory, there is a unique correspondence between $p(t)$ and $\varphi(t)$. If $\tilde{p}(s)$ is the Laplace transform of $p(t)$, then [14]

$$\frac{1}{\tilde{p}(s)}=s+\int_0^{\infty}(1-e^{-st})\varphi(t)dt\ . \tag{3.4}$$

Hence a regenerative event can as well be specified by its $p(t)$, also called the Kingman's p function.

Of central importance is the stochastic function $U(t)$, the total number of jump events in $(0,t)$, which may be expressed as

$$U(t)=\int_0^t Z(t')dt'\ , \tag{3.5}$$

where $Z(t)$ is the indicator function for the jump event, taking values 0 and 1. Obviously,

$$p(t)=\Pr[Z(t)=1],\quad \langle U(t)\rangle=\int_0^t p(t')dt'\ . \tag{3.6}$$

In order to obtain the asymptotic behaviors of $p(t)$ as $t\to\infty$, we apply Eq. (3.4) to $\varphi(t)$ of the KBS model [Eq. (3.2)], and expand for small s, which yields

$$\frac{1}{\tilde{p}(s)}=\begin{cases}A\Gamma(1-\alpha)s^{\alpha}+O(s), & \alpha<1\\ As\ln s+O(s), & \alpha=1\\ (1+\tau_1)s-A|\Gamma(1-\alpha)|s^{\alpha}+O(s^2), & 1<\alpha<2\\ (1+\tau_1)s-As^2\ln s+O(s^2), & \alpha=2\\ (1+\tau_1)s-(\sigma^2/2)s^2+O(s^{\alpha},s^3), & 2<\alpha\end{cases} \tag{3.7}$$

where τ_1 is the mean, and σ^2 the variance, of the interjump interval. Three cases are distinguished in Eq. (3.7), according to (a) $\alpha>2,\tau_1,\sigma<\infty$; (b) $1<\alpha\le 2$, $\tau_1<\infty$, $\sigma=\infty$; and (c) $\alpha\le 1$, $\tau_1=\sigma=\infty$.

Using the Tauberian theorems [28], one obtains

$$p(t)\underset{t\to\infty}{\sim}\begin{cases}(\sin\alpha\pi/A\pi)t^{\alpha-1}, & \alpha<1\\ \dfrac{1}{A\ln(t/A)}, & \alpha=1\\ \dfrac{1}{(1+\tau_1)}, & 1<\alpha\ .\end{cases} \tag{3.8}$$

Equation (3.8) tells us that if $\alpha>1$, the probability of observing a recurrence of the jump eventually becomes independent of time. By contrast, for $\alpha \leq 1$, this probability decreases and vanishes for $t \to \infty$. Hence, as time elapses, longer and longer steps will have a chance to occur, and the jump event will seem progressively rarer. However, one should not be misled into thinking that the process is transient, because after an arbitrarily long time the event will recur with probability 1, when the whole process restarts anew (regenerates).

Combining Eqs. (3.6) and (3.8) the mean value of $U(t)$ yields

$$\langle U(t)\rangle \sim \begin{cases} (\sin\alpha\pi/A\alpha\pi)t^{\alpha}, & \alpha<1 \\ \dfrac{t}{A\ln(t/A)}, & \alpha=1 \\ t/(1+\tau_1), & 1<\alpha\ . \end{cases} \tag{3.9}$$

Therefore, for $\alpha<1$, the regenerative events occur on a random fractal subset in time [29], with a fractal dimension $d_H^t=1$ if $\alpha\geq 1$, and α if $\alpha<1$.

C. Non-Gaussian fluctuations and the Lévy distributions

The inverse stochastic function $T(u)$ of $U(t)$ is the time necessary for $U(t)$ to attain the value u. Obviously,

$$\Pr[U(t)\geq u]=\Pr[T(u)\leq t]\ , \tag{3.10}$$

from which one can deduce [14]

$$\langle \exp[-sT(u)]\rangle = \exp[-u/\tilde{p}(s)]\ . \tag{3.11}$$

The moments of $T(u)$ can be easily calculated from this formula together with Eq. (3.7). For instance, for $\alpha>2$, we have

$$\langle T(u)\rangle \underset{u\to\infty}{\sim} u(1+\tau_1),\quad \mathrm{var}(T(u))\sim u\sigma^2\ . \tag{3.12}$$

Since $T(u)$ is a sum of time intervals of independent steps, its ergodic properties and fluctuations can be expressed using limit theorems for stationary processes with independent non-negative increments [28]. And using Eq. (3.10) statements about $U(t)$ may be obtained. For instance, if $\alpha>2$, so that $\sigma<\infty$, the central limit theorem asserts that

$$\lim_{u\to\infty}\Pr\left[\frac{T(u)-u(1+\tau_1)}{\sqrt{u}\,\sigma}\leq x\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-y^2/2}dy\ . \tag{3.13}$$

By inverting the equation $t=\sqrt{u}\,x\sigma+u(1+\tau_1)$ for $u\to\infty$, and using Eq. (3.10), one concludes that $\Pr[U(t)\geq t/(1+\tau_1)-x\sigma t^{1/2}/(1+\tau_1)^{2/3}]$ converges to the Gauss distribution.

A similar argument can be carried out for $\alpha\leq 2$, but the limiting law will be of Lévy type. Let $G_\alpha(x)$ denote the asymmetric Lévy stable distributions, with the Fourier transform $\tilde{g}_\alpha(z)$ of its density $g_\alpha(x)$ given by [30]

$$\ln\tilde{g}_\alpha(z)=\begin{cases} -|z|^{\alpha}\Gamma(1-\alpha)[\cos(\pi\alpha/2)-i\,\mathrm{sgn}(z)\sin(\pi\alpha/2)] & \text{if } 0<\alpha<2,\ \alpha\neq 1\ , \\ -|z|[\pi/2+i\,\mathrm{sgn}(z)\ln|z|] & \text{if } \alpha=1\ , \\ -|z|^2/2 & \text{if } \alpha\geq 2\ . \end{cases} \tag{3.14}$$

One can prove the following limit theorems:

$$G_\alpha(x)=\lim_{t\to\infty}\begin{cases} \Pr\left[U(t)\geq \dfrac{t}{1+\tau_1}-x\dfrac{\sigma t^{1/2}}{(1+\tau_1)^{3/2}}\right], & \alpha>2 \\ \Pr\left[U(t)\geq \dfrac{t}{1+\tau_1}-x\dfrac{(At\ln t)^{1/2}}{(1+\tau_1)^{3/2}}\right], & \alpha=2 \\ \Pr\left[U(t)\geq \dfrac{t}{1+\tau_1}-x\left[\dfrac{At}{(1+\tau_1)^{\alpha+1}}\right]^{1/\alpha}\right], & 1<\alpha<2 \\ \Pr\left[U(t)\geq \dfrac{t}{A\ln(t/A)}-x\dfrac{t}{A[\ln(t/A)]^2}\right], & \alpha=1 \\ \Pr\left[U(t)\geq \dfrac{t^{\alpha}}{Ax^{\alpha}}\right], & \alpha<1\ . \end{cases} \tag{3.15}$$

These expressions are consistent with more formal statements in Ref. [31].

It is not straightforward to derive quantities such as the variance of $U(t)$, from the limiting distributions. Fortunately, a formula which was derived by Kendall based on Eq. (3.11) [32] expresses the Laplace transform of any moment of $U(t)$ in terms of that of the Kingman's p function, $\tilde{p}(s)$, or explicitly,

$$\int_0^{\infty}e^{-st}\langle U^k(t)\rangle dt=\frac{k!}{s[\tilde{p}(s)]^k}\ . \tag{3.16}$$

Hence, using again the Tauberian theorem, with the aid of Eq. (3.7), one obtains

$$\mathrm{var}(U(t)) = \langle U^2(t) \rangle - \langle U(t) \rangle^2$$

$$\sim \begin{cases} t, & \alpha > 2 \\ t \ln t, & \alpha = 2 \\ t^{3-\alpha}, & 1 < \alpha < 2 \\ \dfrac{t^2}{(\ln t)^2} o(1), & \alpha = 1 \\ t^{2\alpha}, & \alpha < 1. \end{cases} \tag{3.17}$$

D. Anomalous diffusion

Because the direction of each displacement step is randomly chosen, the total displacement after a time t is in average a "sum" of individual displacements, with $U(t)$ "summands," thus one can write

$$\langle \mathbf{r}^2(t) \rangle \sim \langle U(t) \rangle \langle \mathbf{r}^2 \rangle_t \sim t^\eta , \tag{3.18}$$

where $\langle \mathbf{r}^2 \rangle_t$ is the mean-squared displacement of a single step, *conditioned* by the fact that the latter cannot last, at most, longer than the time t. This implies that

$$\langle \mathbf{r}^2 \rangle_t = \int \mathbf{r}^2 \Psi(\mathbf{r}, t') \Theta(t - t') d\mathbf{r}\, dt' \tag{3.19}$$

with $\Theta(x)$ being a Heaviside function. A direct calculation yields

$$\langle \mathbf{r}^2 \rangle_t \underset{t \to \infty}{\sim} \begin{cases} \text{const}, & \alpha > 2\nu \\ \ln t, & \alpha = 2\nu \\ t^{2\nu - \alpha}, & \alpha < 2\nu . \end{cases} \tag{3.20}$$

Combining Eq. (3.18) with Eqs. (3.9) and (3.20), one concludes that (i) if $\alpha > 1$,

$$\langle \mathbf{r}^2(t) \rangle \sim \begin{cases} t, & \alpha > 2\nu \\ t \ln t, & \alpha = 2\nu \\ t^{2\nu + 1 - \alpha}, & \alpha < 2\nu . \end{cases} \tag{3.21}$$

(ii) if $\alpha = 1$,

$$\langle \mathbf{r}^2(t) \rangle \sim \begin{cases} \dfrac{t}{\ln t}, & \alpha > 2\nu \\ t, & \alpha = 2\nu \\ \dfrac{t^{2\nu}}{\ln t}, & \alpha < 2\nu . \end{cases} \tag{3.22}$$

and (iii) if $\alpha < 1$,

$$\langle \mathbf{r}^2(t) \rangle \sim \begin{cases} t^\alpha, & \alpha > 2\nu \\ t^\alpha \ln t, & \alpha = 2\nu \\ t^{2\nu}, & \alpha < 2\nu . \end{cases} \tag{3.23}$$

These results are in complete agreement with those originally derived by Klafter, Blumen, and Shlesinger [12], who used a technique of Laplace transform of probability distributions. The new derivation seems more transparent, and demonstrated the important role played by the random function $U(t)$.

E. Space-time fractals

In this simple model of stochastic diffusion, space and time are coupled in an explicit way, namely a jump (a constant step) in time is uniquely associated with a "stopover" (straight trajectory) in space. We have shown that the ensemble of the time instants where jump events occur form a fractal set if $d_H^t = \alpha \le 1$. When do the stopover points form a fractal object in space? Let d_H^r denote the fractal dimension of the latter. The "mass" of the stopover points is given by $U(t)$ for a finite t. Since this mass is covered in average by a spatial extension of $\langle \mathbf{r}^2(t) \rangle^{1/2}$, we have

$$\langle U(t) \rangle \sim \langle \mathbf{r}^2(t) \rangle^{d_H^r/2} \quad \text{or } \eta = 2 d_H^t / d_H^r . \tag{3.24}$$

It is known that for a Brownian motion a particle's trajectory would fill up a two-dimensional surface and we have $d_H^r = 2$. We shall speak of a fractal in space only in the restricted sense that the stopover points form a set in space with a fractional dimension ($d_H^r < 2$).

From Eq. (3.24) some general statements may be drawn. Hence the normal diffusion is related to the absence of a fractal both in space and time ($d_H^t = 1$ and $d_H^r = 2$ imply $\eta = 1$), as well as to the exceptional cases when a spatial fractal and a temporal one are connected via $2 d_H^t = d_H^r < 2$. Perhaps somewhat surprising is the observation that fractal time *reduces* the transport pace: if a fractal is only present in time the transport is always subdiffuse ($d_H^t < 1$ and $d_H^r = 2$ imply $\eta < 1$). In this case the diffusion coefficient D defined as

$$D = \lim_{t \to \infty} \frac{\langle \mathbf{r}^2(t) \rangle}{2\, t d} \tag{3.25}$$

vanishes. The supradiffusive transport (with $D = \infty$) is always conditioned by the presence of a fractal in space ($d_H^r < 2$ is necessary, and $d_H^r < 2 d_H^t$ is sufficient, to imply $\eta > 1$).

Applying Eq. (3.24) to the KBS model with the aid of Eqs. (3.21)–(3.23) and Eq. (3.9), one concludes that if $\alpha \ge 1$,

$$d_H^r = \begin{cases} 2 & \text{if } \alpha \ge 2\nu \\ \dfrac{2}{2\nu + 1 - \alpha} & \text{if } \alpha < 2\nu \end{cases} \tag{3.26}$$

and if $\alpha < 1$,

$$d_H^r = \begin{cases} 2 & \text{if } \alpha \ge 2\nu \\ \dfrac{\alpha}{\nu} & \text{if } \alpha < 2\nu . \end{cases} \tag{3.27}$$

These results are consistent with the above general conclusions, and suggest a classification of the four transport regimes summarized in Eqs. (3.21)–(3.23) in terms of the presence or absence of a fractal in time or/and in space. In particular, one sees that $d_H^r = 2$ can still be the case even with an anomalous diffusion behavior, if a fractal is realized only in time (when $2\nu < \alpha < 1$). On the other hand, regardless whether a fractal is present in time, the condition under which $d_H^r < 2$ coincides with $\alpha < 2\nu$, that is, when the single-step displacement is divergent, $\langle \mathbf{r}^2 \rangle_t \to_{t\to\infty} \infty$ [Eq. (3.20)], and therefore there is a lack of a characteristic scale *in space*.

IV. VELOCITY FUNCTION

A. Velocity distribution and normalizability

The velocity $\mathbf{v}(t)$ of a Lévy particle is constant during straight motion steps, and exhibits discontinuous changes at jump time instants (see Fig. 1 for illustrations). Its values at two time instants t_1 and t_2 are statistically independent if a jump event occurred at a time between t_1 and t_2. The probability distribution of a single-step velocity is directly given by $\Psi(\mathbf{r},t)$ and $\mathbf{v}=\mathbf{r}/t$, which yields

$$\Psi(\mathbf{v},t)=\varphi(t)\delta(v-t^{\nu-1}), \quad \varphi(t) \underset{t\to\infty}{\sim} at^{-(\alpha+1)} . \tag{4.1}$$

Note that, for $\nu>1$ ($\nu<1$), the longer is one constant

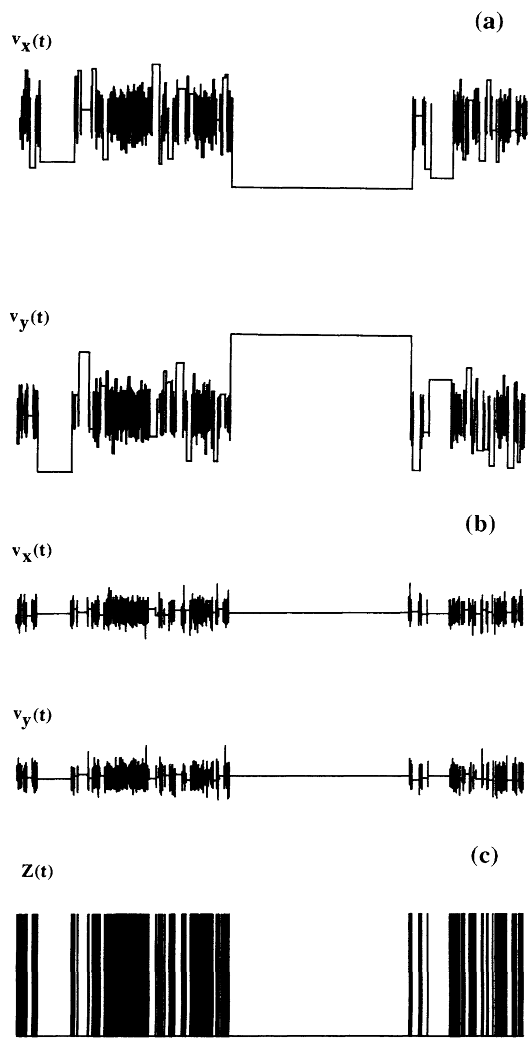

FIG. 1. Velocity function of a two-dimensional KBS model with $\alpha=0.8$. (a) $\nu=0.8$ and (b) $\nu=1.2$. A same sample path of 3000 times steps was chosen for (a) and (b), which thus differ only by the value of ν. In (c) is shown the corresponding indicator function $Z(t)$ for the jump event, where a self-similar structure is apparent.

step, the larger (the *smaller*) is the velocity during that step (cf. Fig. 1). By a coordinate change, and according to $\Psi(\mathbf{v},t)d\mathbf{v}=\Psi(v,t)dv$, one can write

$$\Psi(v,t)=\psi(v)\delta(t-v^{1/(\nu-1)}),$$
$$\psi(v)\sim v^{-[\alpha/(\nu-1)+1]} \quad \text{for} \quad \begin{cases} v\to\infty & \text{if } \nu>1 \\ v\to 0 & \text{if } \nu<1 \end{cases} \tag{4.2}$$

and $\psi(\mathbf{v})=\psi(v)v^{1-d}$.

Now, the probability to observe a value of v at any time is the probability of having a constant step with that value of v, times the duration of the time step. Therefore the probability density of v which would be invariant in time, say $\rho(v)$, satisfies

$$\rho(v)=v^{1/(\nu-1)}\psi(v)\sim v^{-[(\alpha-1)/(\nu-1)+1]}$$
$$\text{for} \quad \begin{cases} v\to\infty & \text{if } \nu>1 \\ v\to 0 & \text{if } \nu<1 \; . \end{cases} \tag{4.3}$$

And $\rho(\mathbf{v})=\rho(v)v^{1-d}$.

From Eq. (4.3) one concludes that the invariant density $\rho(v)$ is normalizable if $\alpha>1$, with the normalizing factor being proportional to τ_1, the mean single-step duration. On the other hand, for $\alpha\le 1$, $\rho(v)$ is not normalizable. This is concomitant with the fact that for $\alpha\le 1$, the decay of $\varphi(t)$ is so slow that τ_1 diverges, and constant steps of extremely long durations dominate the dynamics. From an observational viewpoint, however, if a physical observation is limited to a time span T, the distribution $\varphi(t)$ ought to be truncated at T; and the velocity distribution $\rho(v)$ ought to be truncated at $v^*=T^{\nu-1}$. The *conditional* probability density

$$\rho(v\,|\,T)=\begin{cases} \dfrac{\rho(v)}{\int_0^{T^{\nu-1}}\rho(v')dv'}, & \nu>1 \\[2ex] \dfrac{\rho(v)}{\int_{T^{\nu-1}}^{\infty}\rho(v')dv'}, & \nu<1 \end{cases}$$
$$\underset{T\to\infty}{\sim} \begin{cases} \rho(v) & \text{if } \alpha>1 \\ \rho(v)/\ln T & \text{if } \alpha=1 \\ \rho(v)T^{\alpha-1} & \text{if } \alpha<1 \end{cases} \tag{4.4}$$

will always be well defined.

An alternative way to interpret this peculiar feature is to consider the probability density $\rho(v,\tau,t)$ for observing a value of v at time t, which is prior to a jump event by time τ. We have

$$\rho(v,\tau,t)=p(t-v^{1/(\nu-1)}+\tau)\psi(v)\Theta(v^{1/(\nu-1)}-\tau) \; , \tag{4.5}$$

where $p(t)$ is the Kingman's p function. For $\alpha>1$, $p(t)$ tends to a constant [Eq. (3.8)], and $\rho(v)$ is recovered as $\rho(v)=\lim_{t\to\infty}\rho(v,t)$, with $\rho(v,t)=\int\rho(v,\tau,t)d\tau$. For $\alpha\le 1$, on the other hand, $p(t)$ asymptotically vanishes in the same manner as the dependence of $\rho(v\,|\,T)$ on T.

In conclusion, the probability density for the velocity of a Lévy particle is not normalizable, if $\alpha\le 1$ when $\tau_1=\infty$ and the jump events occur rarely enough to form only a fractal subset in time. In that case the conditional probability Eqs. (4.4) are to be used in statistical calculations, e.g., of the autocorrelation function of $\mathbf{v}(t)$.

B. Velocity autocorrelation and transport behaviors

The mean displacement of a particle can be expressed in terms of the correlation function $C(t)$ of its velocity, as follows:

$$\begin{aligned}\langle \mathbf{r}^2(t)\rangle &= \int\int_0^t dt'dt''\langle \mathbf{v}(t')\mathbf{v}(t'')\rangle \\ &= 2t\int_0^t dt' C(t') - 2\int_0^t t'dt' C(t') \ . \end{aligned} \tag{4.6}$$

As a consequence, if $C(t) \underset{t\to\infty}{\sim} t^{-\gamma}$, then

$$\langle \mathbf{r}^2(t)\rangle \sim t^{\eta}, \quad \eta = \begin{cases} 1 & \text{if } \gamma > 1 \\ 2-\gamma & \text{if } \gamma < 1 \ . \end{cases} \tag{4.7}$$

The velocity correlation function of a Lévy motion, for $\alpha > 1$, is given by

$$C(t) = \langle \mathbf{v}(0)\mathbf{v}(t)\rangle = \int d\mathbf{v}\int d\mathbf{v}'\rho(\mathbf{v})p(\mathbf{v}';t|\mathbf{v})\mathbf{v}\mathbf{v}' \ , \tag{4.8}$$

where $p(\mathbf{v}';t|\mathbf{v};0)$ is the transition probability which takes the form

$$p(\mathbf{v}';t|\mathbf{v}) = \begin{cases} \delta^d(\mathbf{v}-\mathbf{v}') & \text{if } \mathbf{v} \text{ and } \mathbf{v}' \text{ belong to a same step} \\ \rho(\mathbf{v}',t) & \text{otherwise} \ . \end{cases} \tag{4.9}$$

If $\mathbf{v}(0)$ and $\mathbf{v}(t)$ do not belong to a same step, the average of their product in Eq. (4.8) would be zero because of the orientation randomization at each jump. Hence nonzero contribution to $C(t)$ comes only from single steps of duration larger than t, i.e., $v > t^{\nu-1}$. Consequently, the velocity correlation function $C(t)$ is simply

$$C(t) = \int_{t^{\nu-1}}^{\infty} dv\, v^{2-[(\alpha-1)/(\nu-1)+1]} \underset{t\to\infty}{\sim} t^{-\gamma} \ , \tag{4.10}$$

with $\gamma = \alpha + 1 - 2\nu$. Applying Eq. (4.7) to Eq. (4.10) one concludes that, for $\alpha > 1$,

$$\eta = \begin{cases} 1 & \text{if } \alpha > 2\nu \\ 2\nu + 1 - \alpha & \text{if } \alpha < 2\nu \ . \end{cases} \tag{4.11}$$

For $\alpha < 1$, the same procedure can be repeated, except that now using the conditional probability $\rho(\mathbf{v},t|T)$ yields an extra factor $\sim T^{\alpha-1}$ in the final expression in Eq. (4.10). Identifying T with the last observed time t, one obtains, for $\alpha < 1$, $\gamma = 2(1-\nu)$ and

$$\eta = \begin{cases} \alpha & \text{if } \alpha > 2\nu \\ 2\nu & \text{if } \alpha < 2\nu \ . \end{cases} \tag{4.12}$$

In this way one recovers once again the four transport regimes. It is to be noted that from the general equation (4.7), $\eta > 2$ only if $\gamma < 0$, i.e., when the velocity autocorrelation *increases* in time.

The power spectrum of a Lévy velocity is obtained from its autocorrelation, using again the Tauberian theorem, which yields

$$S(\omega) \underset{\omega\to 0}{\sim} \omega^{-\beta} \ , \quad \beta = \begin{cases} 2\nu - \alpha & \text{if } \alpha > 1 \\ 2\nu - 1 & \text{if } \alpha < 1 \ . \end{cases} \tag{4.13}$$

Equation (4.13) shows that there is an infrared divergence ($1/f$-like noise), if $2\nu > \alpha > 1$, or $\alpha < 1$ and $\nu > 1/2$.

C. ϵ entropy of the velocity function

Let velocity signals be digitalized by a scale of ϵ, and the time discretized by τ. One can define a Shannon-like entropy $H(\epsilon,\tau,T)$, as function of ϵ and for a time span $T = n\tau$, as the average of $-\ln \Pr(\epsilon,\tau,n)$, where $\Pr(\epsilon,\tau,n)$ is the probability for a discretized velocity sample of length n [33]. For a Lévy motion, which consists of statistically independent steps, $\Pr(\epsilon,\tau,n)$ can be written as a product of identical distributions:

$$\Pr(\epsilon,\tau,n) = \rho_\epsilon(\mathbf{v}_0)\prod_{i=1}^{N_n}\psi_\epsilon(\mathbf{v}_i) \ , \tag{4.14}$$

where $\rho_\epsilon(\mathbf{v})$ and $\psi_\epsilon(\mathbf{v})$ are the discrete counterparts of $\rho(\mathbf{v})$ and $\psi(\mathbf{v})$, respectively, and N_n is the number of jump events during n time units. Thus we have

$$\langle -\ln \Pr(\epsilon,\tau,n)\rangle \sim \langle N_n\rangle\langle \ln^{-1}\psi_\epsilon(\mathbf{v})\rangle \ . \tag{4.15}$$

The discrete form of $\varphi(t)$ is [cf. Eq. (3.2)]

$$\varphi(k,\tau) \sim \tau\varphi(k\tau) \sim a'k^{-(\alpha+1)} \ , \tag{4.16}$$

where $a' = a\tau^{-\alpha}$. The discrete analogy to Eq. (3.9), due to Feller [13], asserts that

$$\langle N_n\rangle \sim \begin{cases} \dfrac{n}{\tau_1'} \sim T & \text{if } \alpha > 1 \\ \dfrac{n}{a'\ln(n/a')} \sim T/\ln T & \text{if } \alpha = 1 \\ \dfrac{n^\alpha}{a'} \sim T^\alpha & \text{if } \alpha < 1 \ , \end{cases} \tag{4.17}$$

where τ_1' is the first moment of $\varphi(k,\tau)$, $\tau_1' = \sum_k k\varphi(k,\tau) \simeq \tau_1/\tau$. Therefore $\langle N_n\rangle$ only depends on T, not on τ.

Furthermore, one can readily show that the second factor on the right-hand side of Eq. (4.15) yields $\sim d\ln(1/\epsilon)$. Combining with Eq. (4.17) we conclude

$$H(\epsilon,T) \sim \begin{cases} T\ln(1/\epsilon)^d & \text{if } \alpha > 1 \\ (T/\ln T)\ln(1/\epsilon)^d & \text{if } \alpha = 1 \\ T^\alpha \ln(1/\epsilon)^d & \text{if } \alpha < 1 \ . \end{cases} \tag{4.18}$$

From Eq. (4.18) it follows that the velocity of a Lévy particle is random if $\alpha > 1$, and sporadic if $\alpha \le 1$, in the sense of Eq. (2.16). Note that the Lévy motion as defined in the KBS model differs also from a Brownian motion in that $h(\epsilon)$ diverges with $1/\epsilon$ logarithmically, rather than polynomially [cf. Eq. (2.15)]. This is related to the fact that $\langle N_n \rangle$ remains a finite quantity for fixed T and arbitrarily small τ, and $H(\epsilon,T)$ is the product of $\langle N_n \rangle$ with an ϵ entropy of a d-dimensional entropy which typically behaves as $d\ln(1/\epsilon)$ [cf. Eq. (2.12)].

V. POSSIBLE EXAMPLES

Anomalous diffusion has been a focus of recent attention [11]. It would be of interest to establish if the KBS scheme [12] for the Lévy motion is realized in some of the physical systems known to exhibit anomalous diffusion, and to seek conditions under which the sporadicity is expected. Let us mention a few possible examples from *deterministic systems.*

Periodic Lorentz gas without horizon. In a periodic Lorentz gas (or extended Sinai billiard) a point mass moves with constant velocity between elastic collisions with fixed hard spheres in a regular lattice. This problem of interest to the foundation of statistical mechanics has been rigorously analyzed and many of its ergodic properties are known [7]. If the interscatter distance is large compared with the scatter radius, the length of free paths is unbounded and the billiard is said to be without horizon. In the case of a cubic lattice which is always without horizon, from an ergodic consideration the probability density for the free path length was shown to be [34]

$$\psi(l)\sim l^{-3} \tag{5.1}$$

hence the corresponding parameter in the KBS model is $\mu^*=3$. With $\nu=1$ one deduces $\alpha=\nu\mu^*-1=2$. Then, the KBS model predicts that the mean-square displacement is [Eq. (3.21)]

$$\langle \mathbf{r}^2(t)\rangle \sim t\ln t \tag{5.2}$$

and the correlation function and the power spectrum for the velocity are, respectively [Eqs. (4.10) and (4.13)],

$$C(t)\sim t^{-1},\quad S(\omega)\sim \ln\frac{1}{\omega}\ . \tag{5.3}$$

Both these predictions agree with the results derived from different arguments and with the numerical simulations [35]. Thus the extended Sinai billiard seems to provide an example of the Lévy motion as described by the KBS model, in spite of the fact that the dynamics is intrinsically deterministic (e.g., the orientation change at each collision, instead of being randomly chosen, obeys the rule "the incidence angle equals the reflection angle").

Chaotic diffusion in regular fields. Particle motion in an ideal, incompressible and time-independent flow possessing the Beltrami property $\mathbf{v}\times\mathrm{rot}\mathbf{v}=0$ can display chaotic dynamics in physical space. Examples for this so-called Lagrangian turbulence are the well-known periodic *ABC* flow [36], and its generalization called the Q flow with quasiperiodic symmetry [37]. There is some numerical evidence that the trajectory of a particle consists of straight steps between random turns that are reminiscent of a Lévy motion, and the transport by chaotic streamlines shows signs of an anomalous diffusion [38].

A related class of systems is the particle motion in a two-dimensional smooth periodic potential which have also been shown to display anomalous diffusion with Lévy characteristics [39]. This and the streamline problem can both be described in a Hamiltonian formalism, and it is possible that the anomalous transport is related to the existence of a hierarchy of cantori [broken Kolmogorov-Arnold-Moser (KAM) invariant tori], which is believed to naturally imply a long tail of the velocity autocorrelation [40,27]. No satisfactory theory is yet available which would allow one to establish a waiting-time distribution like Eq. (3.2) and derive analytically the exponent of its long tail. In particular, we do not know if sporadic behavior can be realized in such systems.

Deterministic maps. The Pomeau-Manneville map [Eq. (1.1)] can be suitably deformed to allow motions between adjacent unit intervals in an one-dimensional array of unit cells. Such a variant was introduced to approximate the phasic diffusion in resistively shunted Josephson junctions [41]. The mean-square displacement was found to be [41,42]

$$\langle r^2(t)\rangle \sim \begin{cases} t, & \alpha>2 \\ t\ln t, & \alpha=2 \\ t^{3-\alpha}, & 1<\alpha<2 \\ t^2/\ln t, & \alpha=1 \\ t^2, & \alpha<1 \end{cases} \tag{5.4}$$

where $\alpha=1/(z-1)$ as in the Pomeau-Manneville case. These results are remarkably consistent with the KBS model [compare with Eqs. (3.21)–(3.23), assuming $\nu=1$]. Here the analogy can be done with certain confidence, because the distribution of the regular ("laminar") phases are known to exhibit an algebraic decay, $\varphi(t)\sim t^{-(\alpha+1)}$. Therefore we expect that the system is sporadic if $\alpha\le 1$, or $z\ge 2$ which includes the usual physical values $z=2,3$. Note, however, that the time here is discrete rather than continuous.

Relative turbulent diffusion. It has been suggested that the relative diffusion of particles in a developed turbulent fluid may be modeled by a Lévy motion, though no evidence is known for a waiting-time distribution with an algebraic decay like Eq. (3.2) [43]. Let us denote by $\mathbf{v}(r)$ the velocity difference of a pair of particles separated by a distance of r. In a scaling regime the classical Kolmogorov law asserts that $\langle|\mathbf{v}(r)|\rangle\sim r^{1/3}$, thus in the sense of a *statistical average* a characteristic time t is associated with a characteristic spatial length r as $r\sim t^{\nu}$ with an exponent $\nu=3/2$. Moreover, from the Richardson law of the relative diffusion $\langle \mathbf{r}^2(t)\rangle\sim t^3$, one has $\eta=3$. It follows that $\alpha\le 1$. However, this does not imply that the relative velocity should be sporadic in time, or that its time course would display very long quiescent phases. In this case, during each characteristic step the relative velocity is not constant but very random in time.

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[29] Recall that each stochastic realization in a time interval $(0,t)$ contains only a finite number of jump events, which thus cannot form a fractal set of uncountably many time instants. Here the term "fractal" is used to describe the self-similar property of the statistical *average* of $U(t)$. Note also that one can apply the Mandelbrot approach discussed in Sec. II to the KBS model, by inserting the expression for $\varphi(t)$ of Eq. (3.2) into Eq. (2.5). One sees that $\phi(t)$ is not normalizable if $\alpha \leq 1$, the same condition for the realization of a fractal in time. We shall see that in this particular model and for the velocity function, our definition of sporadicity [Eq. (2.16)] coincides with that of Mandelbrot.

[30] Except for a few special values of α, the Lévy distribution density $g_\alpha(x)$ does not have a closed form. Their general properties, however, are well known. The asymmetric laws as defined by the Fourier transform [Eq. (3.14)] are unimodal, and satisfy the following asymptotic form: (1) for $2>\alpha>1$,

$$g_\alpha(x)\sim\begin{cases}x^{-(1+\alpha)} & \text{if } x\to+\infty\\ \exp(-c|x|^{\alpha/(\alpha-1)}) & \text{if } x\to-\infty\end{cases}$$

with $c>0$. (2) For $\alpha=1$,

$$g_1(x)\sim\begin{cases}x^{-2} & \text{if } x\to+\infty\\ \exp[-\exp(-x)] & \text{if } x\to-\infty\end{cases}$$

and (3) for $\alpha<1$,

$$g_\alpha(x)\sim\begin{cases}x^{-(1+\alpha)} & \text{if } x\to+\infty\\ \exp(-cx^{1/(\alpha-1)}) & \text{if } x\to0+\end{cases}$$

with $c>0$, and $g_\alpha(x)\equiv0$ for $x<0$. See V. Zotolarev, *One-Dimensional Stable Distributions,* Translations of Mathematics Monographs Vol. 65 (American Mathematical Society, Providence, RI, 1986).

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[33] Since $\mathbf{v}(t)$ is discontinuous there is no direct connection between the time step size τ and the discretization size ϵ, unlike the Ornstein-Uhlenbeck process which is mean-square continuous with a Hölder exponent 1/2.

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