SPONTANEOUS ACTIVITY IN A LARGE NEURAL NET: BETWEEN CHAOS AND NOISE

Xiao-Jing Wang

Mathematical Research Branch
National Institutes of Health
Bethesda, Maryland 20892, USA

In what follows I would like to discuss the intrinsic oscillatory dynamics of a large network of neurone-like analog units. Using an appropriate entropic quantity called the ϵ-entropy, the spontaneous activity of the net will be shown to be intermediate between low-dimensional chaos and random noise. This kind of network dynamics seems to reflect some aspects of the ground-state fluctuations of the mammalian brain systems during wakefulness.

Let us consider a model network given by

$$\tau_0 dv_i/dt = -v_i + \sum_{j=1}^{N} J_{ij}\phi(v_j); \qquad i = 1, 2, ..., N \tag{1}$$

where τ_0 is a time constant, J_{ij} is the connection matrix, and $\phi(x)$ is the input-output function of a sigmoid type, e.g. $\phi(x) = tanh(gx)$ with a gain parameter g. Eq.(1) represents a *bona fide* "connectionist" model: each unit alone would merely relax to its rest $v_i \equiv 0$ in a trivial fashion. When such units are connected by strong enough nonlinear interactions, however, collective dynamics and computational abilities can emerge in the network.

In a Hopfield network (Hopfield 1982, 1984) with *symmetric* connection, $J_{ij} = J_{ji}$, we know that the model Eq.(1) has an analogy to a spin glass (a net of magnetic dipoles with random connections). Then, there exists a global Liapunov function, like the Hamiltonian of a spin glass, which ensures that the time evolution of the net would always converge to a fixed point. Such a fixed point attractor can be interpreted as a content-addressable memory, and a main question is how many such memories can be stored in a net of N units, with desired recall properties. With J_{ij} prescribed according to a Hebb-like learning rule, and at the large gain limit ($g \to \infty$), this problem can be treated analytically by statistical mechanics of spin glass (cf. Amit (1989)).

If the symmetry assumption is relinquished, $J_{ij} \neq J_{ji}$, then an attractor need no longer be steady: the net can oscillate. What kind of spacetime pattern(s) the net would have as attractor(s) depend on the network architecture (Amari 1972). One possibility would be to keep the connections in the all-to-all form, hence disregarding the spatial geometry all together. A concrete model system of this kind, studied recently by Sompolinsky *et al* (1988), corresponds to the case where the connection matrix J_{ij} is assumed to be generated by independent random Gaussian variables, with $[J_{ij}] = 0$, $[J_{ij}^2] = J^2/N$ and $[J_{ij}J_{i'j'}] = 0$ if $i \neq i'$ or $j \neq j'$ (square brackets denote average over the distribution of J_{ij}). I shall henceforth be restricted to this example.

Self-Organization, Emerging Properties, and Learning
Edited by A. Babloyantz, Plenum Press, New York, 1991

Notice that, once J_{ij} is fixed (one says the random interacting links between units are "frozen"), Eq.(1) is a *deterministic* dynamical system. Since the network is not subject to any external stimulation, the dynamics we shall be dealing with will be called its spontaneous activity.

It is easy to see that by a rescaling of v_i and t, Eq.(1) has only one (dimensionless) parameter, gJ. If $gJ = 0$, the solution of Eq.(1) is obviously $v_i \equiv 0$. With increased gJ, this branch of fixed point attractor can become destabilized. Using a time-dependent mean field theory, Sompolinsky *et al* (1988) found that, for an extensively large network, $N \to \infty$, if $gJ < 1$, the only kind of attractor is fixed point. A transition occurs at $gJ = 1$, beyond which the network displays a chaotically oscillatory state (cf. Fig.1). Furthermore, in such a randomly connected network one would not expect any regular spatial pattern or coherence, which would restrict the number of excited degrees of freedom (modes) to a small figure. Consequently, beyond this critical point, the chaotic attractor is immediately "fully developed": it appears to have an infinite dimension, and infinite number of positive Liapunov exponents (Sompolinsky *et al* 1988; Sompolinsky 1990). One observes that this sharp transition is reminiscent of the Manneville-Pomeau intermittency (Pomeau & Manneville 1980) which, as a bifurcation from a *limit cycle* to chaos, describes a universal scenario of "route to chaos" in low dimensional dynamical systems. I intend to discuss this "intermittent transition to spacetime chaos" in a separate communication.

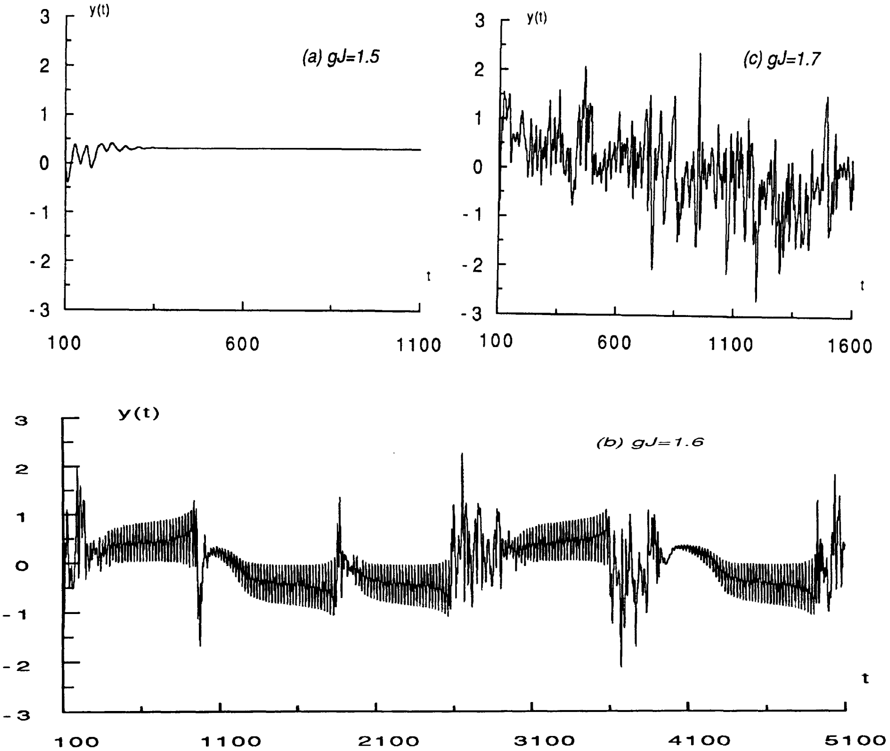

Fig.1. Numerical evidence of intermittency (b) near a transition from a fixed point (a) to a chaotic flow (c). The net Eq.(1) was simulated with $N = 500$ units, J_{ij} were randomly chosen, then fixed. Plotted here are the time evolutions of a single unit for different values of gJ. The intermittent behavior in (b) consists of long regular phases interrupted randomly by chaotic bursts.

Let us focus on this spacetime chaos itself, and attempt to characterize it by its dynamical entropy. Let us recall that, for a finite dynamical system, a chaotic attractor can generate an information flow of finite amount per unit time, quantified by the Kolmogorov-Sinai entropy, h_{KS} (cf. Eckmann & Ruelle 1985; Shaw 1981) This is possible because a *deterministic* chaotic system with *continuous* variables has a correspondence to a *stochastic* process with *discrete* states, via "symbolic dynamics". Then, the Kolmogorov-Sinai entropy is equal to the Shannon entropy per unit time of the associated stochastic process. These discrete states may be viewed as a result of a digitalization of the continuous variables that is *inherent* to the original system. Since the conversion is *exact*, there exits a finite size ϵ_0 of the digitalization accuracy ϵ, beyond which no additional information can be gained. In other words, if one computes the information production rate as a function of ϵ, say $h(\epsilon)$, then it will equal to h_{KS} for all $\epsilon \leq \epsilon_0$.

For a chaotic "flow", with many active degrees of freedom, this Kolmogorov-Sinai entropy is infinite. In such a case one may pretend to associate a deterministic spacetime chaos to a *continuous* variable stochastic process. Indeed, for the network under consideration, Sompolinsky *et al* (1988) showed that, in the limit $N \to \infty$, Eq.(1) is reduced to a dynamical mean-field equation of a single unit, which reads

$$dv_i/dt = -v_i + \eta_i(t), \tag{2a}$$

where $\eta_i(t)$ is a nonMarkovian, Gaussian field, representing the averaged input from other units. Its autocorrelation function

$$< \eta_i(t)\eta_i(t+\tau) >= C(\tau) =< \phi(v_i(t))\phi(v_i(t+\tau)) >, \tag{2b}$$

where $\phi(x) = tanh((gJ)x)$, is evaluated within the mean-field theory (angular brackets denote average with respect to the distribution of $\eta_i(t)$).

The mean-field equation tells us that as $N \to \infty$, the network behaves as if it consisted of *independent* units, embeded in a common field $\eta(t)$. The subscript i becoming unnecessary, will be omitted thereafter.

In case such a connection can be made between a deterministic large system and a probabilistic one, we can introduce (Gaspard & Wang 1990) the notion of ϵ-entropy, initially proposed for continuous stochastic processes (independently by A.N. Kolmogorov and C. Shannon), to deterministic spacetime chaos. The idea of the ϵ-entropy is the following: although the entropy of an analog signal $x(t)$ from a stochastic process is properly speaking infinite, its amount of creating information rate is always bounded, if it is monitored by instruments with finite precision. Suppose the measured signal $y(t)$ differs from the real signal $x(t)$ by a mean quantity, e.g. ϵ is the average of $(x-y)^2$. For instance, $y(t)$ can be a digitalization of $x(t)$ within ϵ. One then can compute the mutual information of x and y, which is well defined. The ϵ-entropy, $h(\epsilon)$, is defined as the minimal mutual information (obtained by optimally choosing the "instrument", or $y(t)$), under the constraint that the error cannot exceed ϵ. For a smaller ϵ, y has to be chosen closer to x, and $h(\epsilon)$ is increased. As $\epsilon \to 0$, $h(\epsilon)$ approaches to the entropy of $x(t)$ itself, i.e. may diverge to infinity. The asymptotic behavior of $h(\epsilon)$ as $\epsilon \to 0$, is a characteristic of the process under consideration.

For a Gaussian stationary process, $h(\epsilon)$ can be written in terms of its power spectrum $f(\omega)$, by the following expressions (due to A.N. Kolmogorov)

$$\begin{aligned} \epsilon^2 &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} min[\theta^2, f(\omega)]d\omega; \\ h(\epsilon) &= \frac{1}{4\pi}\int_{-\infty}^{+\infty} max[0, \log_2 \frac{f(\omega)}{\theta^2}]d\omega. \end{aligned} \tag{3}$$

where $h(\epsilon)$ is parameterized via θ.

This formula implies that the asymptotic form of $h(\epsilon)$ is determined, in the Gaussian cases, by the ultraviolet part of the power spectrum, $f(\omega)$ as $\omega \to \infty$. For instance, an Ornstein-Uhlenbeck process would have a Langevin equation of the same form as Eq.(2), but $\eta(t)$ would be a white noise, $< \eta(t)\eta(t') >= \sigma^2\delta(t-t')$. Then, $f(\omega) = \sigma^2/(2\pi(1+\omega^2))$. From Eq.(3) one can readily show that

$$h(\epsilon) \sim \frac{\sigma^2}{\pi^3}(\frac{1}{\epsilon})^2 \tag{4}$$

That is, the ϵ-entropy of a stationary, Gaussian and Markov process diverges as ϵ^{-2}.

In our case of a large network, the ϵ-entropy is an extensive quantity, proportional to N. In the limit $N \to \infty$, units are essentially decoupled, each being governed by Eq.(2). Thus we can evaluate the ϵ- entropy *per unit*, on the basis of Eq.(2). Since Eq.(2) is linear, and $\eta(t)$ is a Gaussian process, so is $v(t)$. Therefore Eq.(3) is applicable, provided that we know the power spectrum of $v(t)$. A self-consistent equation for the correlation function of $v(t)$ has been given in Sompolinsky *et al* (1988), and solved for $0 < \vartheta \equiv gJ - 1 \ll 1$, yielding

$$\Delta(\tau) \equiv < v(t)v(t+\tau) > \simeq \vartheta cosh^{-2}(\vartheta\tau/\sqrt{3}). \tag{5}$$

The power spectrum $f_v(\omega)$ can be obtained by the Fourier transform of $\Delta(\tau)$,

$$\begin{aligned} f_v(\omega) &= \sqrt{\frac{2}{\pi}}\int_0^\infty cos(\omega\tau)\Delta(\tau)d\tau \\ &= \sqrt{\frac{6}{\pi}}(\omega/\omega_0)sinh^{-1}(\omega/\omega_0) \sim_{\omega\to\infty} \omega/\omega_0 \exp(-\omega/\omega_0). \end{aligned} \tag{6}$$

where $\omega_0 = \frac{2\vartheta}{\sqrt{3}\pi}$.

The fact that $f_v(\omega)$ decreases *exponentially* as $\omega \to \infty$ implies that $v(t)$ is smooth in time. A way to see this is to rewrite $v(t)$ in a "spectral representation" (Yaglom 1962):

$$v(t) = \int_{-\infty}^{+\infty} e^{i\omega t} z(\omega)d\omega, \tag{7a}$$

where $z(\omega)$ are independent Gaussian random variables, with

$$< z(\omega)z^*(\omega') >= f_v(\omega)\delta(\omega - \omega'). \tag{7b}$$

Then, the n-th derivative of $v(t)$ with respect to time t is clearly well defined, for all n, and is also a Gaussian process. Its power spectrum given by $\omega^{2n} \cdot f_v(\omega)$ behaves well because $f_v(\omega)$ decays fast enough for large ω.

Now, applying the Kolmogorov formula Eq.(3) to $f_v(\omega)$, one obtains

$$h(\epsilon) \sim \frac{2\vartheta(\ln 2)^2}{\sqrt{3}\pi^2}(\log_2 \frac{1}{\epsilon})^2. \tag{8}$$

One can draw several interesting conclusions from this brief calculation. On one hand, $h(\epsilon)$ is unbounded. This implies that the spacetime chaos in the neural net Eq.(1) with $gJ > 1$ has an unbounded dynamic entropy, and *a fortiori* of infinite dimension. On the other hand, recalling that ϵ is essentially the accuracy limit by which one monitors the output of a continuous stochastic process, one remarks that for a given, small ϵ, the information production rate per unit time for this system of formal neurones, is qualitatively different from that of a "random noise", such as the Ornstein-Uhlenbeck one (compare Eq.(4) with Eq.(8)). To illustrate our point, let $\vartheta = 0.1$, $\sigma^2 = 0.1$. If $\epsilon = 10^{-3}$, then $h(\epsilon) \sim 0.5$ bits per unit time for the Eq.(2), and $h(\epsilon) \sim 3 \times 10^3$ bits per unit time for the Ornstein-Uhlenbeck process. And if $\epsilon = 10^{-7}$, we have $h(\epsilon) \sim 3$ bits per unit time for the former, and $h(\epsilon) \sim 3 \times 10^{11}$ bits per unit time for the latter case!

The behavior of this network is intermediate between a low dimensional chaos and noise in the following sense: As mentioned above, for a determinstic chaos one would expect

$$h(\epsilon) \sim_{(1/\epsilon)\to\infty} h_{KS}; \quad \text{and} \quad \frac{d}{d(1/\epsilon)}h(\epsilon) \sim_{(1/\epsilon)\to\infty} 0; \tag{9a}$$

while for a Ornstein-Uhlenbeck process one has

$$h(\epsilon) \sim_{(1/\epsilon)\to\infty} \infty; \quad \text{and} \quad \frac{d}{d(1/\epsilon)}h(\epsilon) \sim_{(1/\epsilon)\to\infty} \infty; \tag{9b}$$

and for the network model

$$h(\epsilon) \sim_{(1/\epsilon)\to\infty} \infty; \quad \text{and} \quad \frac{d}{d(1/\epsilon)}h(\epsilon) \sim_{(1/\epsilon)\to\infty} 0. \tag{9c}$$

In other words, even so $h(\epsilon)$ diverges to infinity as $\epsilon \to 0$, its growth rate tends to zero. If it is plotted versus $1/\epsilon$, one would observe an apparent plateau as $1/\epsilon \to \infty$, where the information production rate $h(\epsilon)$ appears seldom sensitive to the change of ϵ.

Of course, the chaotic state we are describing here being the spontaneous activity of the network Eq.(1), one may prefer not to speak of "information flow", if the word is reserved to information that the network can process about its *external world.* Nevertheless, $h(\epsilon)$ is a characteristic of the intrinsic dynamics of the net and, if a random noise is present, it can be used to distinguish the signal (the output of Eq.(2), $v(t)$) from noise. Let us add a white noise to Eq.(2), with a variance $\sigma^2 \ll \vartheta$. Then, the power spectrum of $v(t)$ is merely the sum of its previous part (Eq.(6)) and that of an Ornstein-Uhlenbeck process. When we again compute $h(\epsilon)$, it is straightfoward to see that there exits a cross-over value of ϵ, say ϵ^*, marking the accuracy limit beyond which the noise starts to manifest. ϵ^* can be estimated as

$$\epsilon^{*2} \sim \frac{\sigma^2}{\pi^2\omega_0}(\ln(1/\sigma^2))^{-1}. \tag{10}$$

Thus, ϵ^{*2} differs from the "noise/signal ratio", σ^2/ω_0, by a logarithmic correction. For $\epsilon \gg \epsilon^*$, the signal is dominant, and $h(\epsilon)$ is given by Eq.(8) as before. On the other hand, for $\epsilon < \epsilon^*$, the noise prevails, and $h(\epsilon)$ eventually becomes the same as for a geniune random noise (Eq.(4)). Presumably, when the net is subject to external inputs, a similar approach can be useful to describe the interplay between "real signals" relevant to the external world and the spontaneous fluctuations, using the notion of ϵ-entropy.

To summarize, the spontaneous activity of a network with random connections (Eq.(1)) can be a dynamic state where individual units are effectively uncorrelated, and fluctuate temporally with no apparent regularities. It is distinguished from both a low-dimensional chaos and "noise", as characterized by its own $h(\epsilon)$ form.

If one wishes to ask whether the present discussion could be of any suggestive value to biological nervous systems, one needs to know which conclusions based on Eq.(1) are dependent on the details of the model, and which are not. Some features of the network are likely vulnerable to such changes. For instance, if one adds a cubic *damping* term to the equation of each unit, Eq.(1), the dynamical mean-field theory would lead to an equation similar to Eq.(2),

$$dv/dt = -v - v^3 + \eta(t), \tag{11}$$

where $\eta(t)$ is again a Gaussian field. Now, because Eq.(11) is not linear, $v(t)$ will no longer be Gaussian.

On the other hand, the condition under which the asymptotic behavior of $h(\epsilon)$ as Eq.(8) is expected, seems quite general, namely that the network dynamics must guarantee the smoothness of its solutions: the trajectory of each unit can be chaotic enough to mimic a continuous stochastic process, and yet remains infinitely differentiable, perhaps even analytic.

Acknowledgements

It is my pleasure to thank Prof. John Rinzel for his invaluable support, Prof. Ilya Prigogine for the invitation to this NATO workshop, and Prof. H. Sompolinsky for a conversation. Numerical simulations were carried out at the National Cancer Institute Advanced Scientific Computing Laboratory. The present work is closely related to an ongoing research in collaboration with Dr. Pierre Gaspard.

References

Amari, S.-I., 1972, Characteristics of random nets of analog neuron-like elements, *IEEE Trans. Syst. Man & Cybern.*, SMC-2:643.

Amit, D.J., 1989, *Modeling Brain Function: the world of attractor neural networks*, Cambridge University Press, New York.

Eckmann,J.-P. & Ruelle,D., 1985, Ergodic theory of chaos and strange attractors, *Rev. Mod. Phys.*, 57:617.

Gaspard, P. & Wang, X.-J., 1990, in preparation.

Hopfield,J., 1982, Neural networks and physical systems with emergent collective computational abilities, *Proc. Natl. Acad. Sci. USA*, 79:2554.

Hopfield,J., 1984, Neurons with graded response have collective computational properties like those of two-state neurons, *Proc. Natl. Acad. Sci. USA*, 81:3088.

Pomeau,Y. & Manneville,P., 1980, Intermittent transition to turbulence in dissipative dynamical systems, *Commun. Math. Phys.*, 74:189.

Shaw,R.S., 1981, Strange attractors, chaotic behavior and information flow, *Zeitschrift für Naturforschung*, A36:80.

Sompolinsky, H., Crisanti, A. & Sommers, H., 1988, Chaos in random neural networks, *Phys. Rev. Lett.*, 61:259.

Sompolinsky, H., 1990, personal communication.

Yaglom, A.M., 1962, *An Introduction to the Theory of Stationary Random Functions*, Dover, New York.