In this note, I will be dealing with linear dynamical systems of the form $\dot{x} = Wx$ where $W$ denotes the connectivity matrix of the system. The eigenvector analysis of such systems is familiar. However, when $W$ is a non-normal matrix (i.e. when $W$ has a non-orthogonal set of eigenvectors), an alternative analysis in terms of Schur decomposition may provide a more informative description of the dynamics in such systems. In Schur decomposition, the connectivity matrix $W$ is decomposed as $W = UAU^{-1}$ where $U$ is unitary and $A$ is an upper triangular matrix. Because $U$ is unitary $U^{-1} = U^*$, hence the decomposition can be equally expressed as $W = UAU^*$. I will assume $U$ is a matrix with real-valued entries, hence it can be thought of as an orthogonal matrix and $U^* = U^T$. Moreover, I will further assume that the eigenvalues of $W$ are distinct: $A_{ii} \neq A_{jj}$ for $i \neq j$. The Schur modes, $u_i$, of the system are given by the columns of $U$. If we now project the system onto the $i$-th Schur mode and denote the projection by $y_i = u_i^T x$, we get:

$$u_i^T x = u_i^T UAU^T x$$

$$\dot{y}_i = \sum_{k=1}^{n} A_{i,k} y_k \quad i = 1, \ldots, n$$

where $A_{i,k}$ denotes the $ik$-th entry of $A$. We first note that for $i = n$, the last equation becomes $\dot{y}_n = A_{n,n} y_n$, hence the solution is given by $y_n = K_n \exp(A_{n,n} t)$ where $K_n$ is given by the initial condition of $y_n$. We can then proceed recursively, plugging the solution we get for each Schur mode into the equation for the following Schur mode. For instance, for $i = n - 1$, we have:

$$\dot{y}_{n-1} = A_{n-1,n-1} y_{n-1} + A_{n-1,n} y_n$$

$$= A_{n-1,n-1} y_{n-1} + A_{n-1,n} K_n \exp(A_{n,n} t)$$

where in the last equation I just plugged in the solution for $y_n$. It is straightforward to verify that the solution of the last equation is given by:

$$y_{n-1} = K_{n-1} \exp(A_{n-1,n-1} t) + K_n \frac{A_{n-1,n}}{A_{n,n} - A_{n-1,n-1}} \exp(A_{n,n} t)$$

where the constant $K_{n-1}$ is again determined by the initial condition of $y_{n-1}$. Iterating this process further, a little bit of algebra reveals that for the $i$-th Schur mode, we get a solution like the following:

$$y_{n-i} = \sum_{j=0}^{i} B_{n-j} \exp(A_{n-j,n-j} t)$$

where:

$$B_{n-j} = K_{n-j} \sum_{p \in \varphi(n-j,n-i)} \frac{\mathcal{N}(p)}{\prod_{l=1}^{\mathcal{N}(p)} \frac{A_{p(l),p(l+1)}}{A_{n-j} - A_{p(l)}}}$$

where we denote the set of paths $p$ from the $(n-j)$-th mode to the $n-i$-th mode by $\varphi(n-j,n-i)$ and the number of modes traversed along the path $p$ by $\mathcal{N}(p)$.

As in eigenvector analysis, the case of repeating eigenvalues ($A_{ii} = A_{jj}$ for some $i \neq j$) must be handled separately, but one proceeds in essentially the same way as before. If the eigenvalues are all the same, i.e. $A_{ii} = A$ for all $i$, we
obtain the following solution:

\[ y_{n-i} = \sum_{j=0}^{i} B_{n-j} \frac{t^j}{j!} \exp(At) \]  

where:

\[ B_{n-j} = \sum_{p} K_{p(1)}^{N(p)} \prod_{l=1}^{p(t),p(t+1)} A_{p(t),p(t+1)} \]  

where the summation is over all length-\( j \) paths \( p \) from earlier modes to mode \( n - i \).